

Set inversion involving interval angle measurements

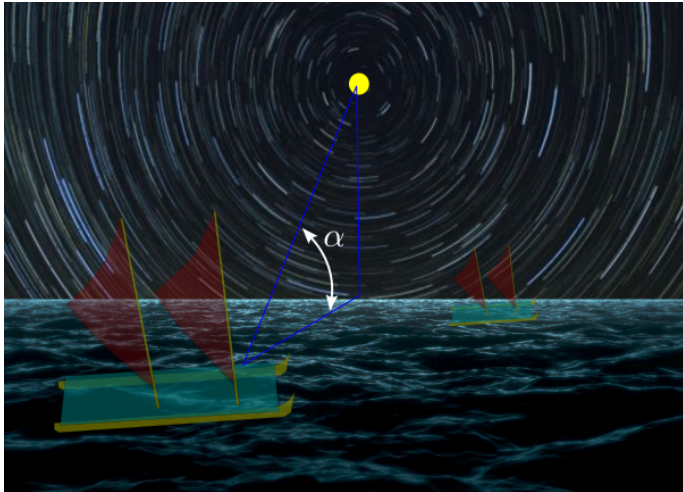
Luc Jaulin



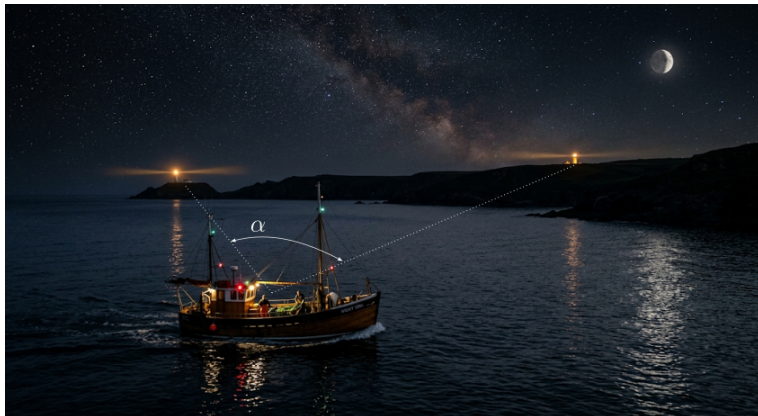
Oldenburg, Germany, October 5, 2026

1. Interval angles





Latitude



Subtended angle



Bearing



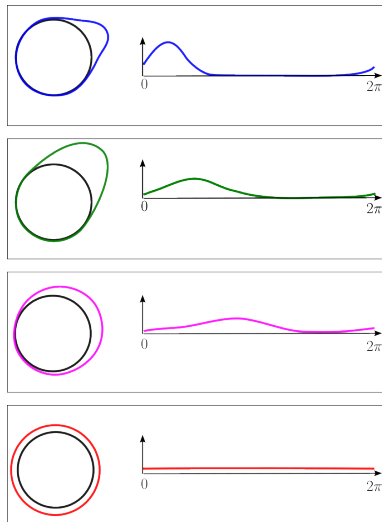
Sextant



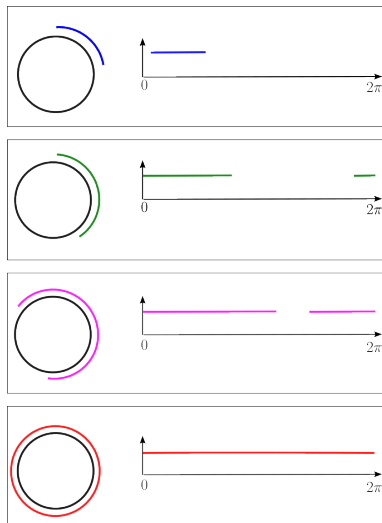
Theodolite



Camera



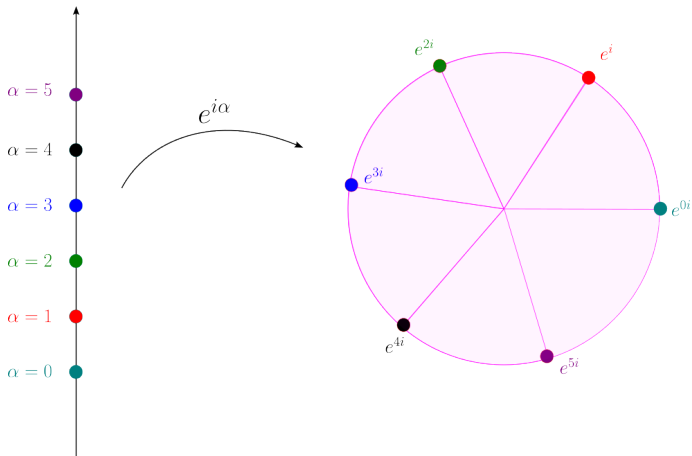
Von mises distributions $\propto \exp(\kappa \cos(\alpha - \bar{\alpha}))$



- An *angle* α is an element of the trigonometric circle S^1 .
- Engineers often represent angles by a real number α
- The complex number associated to α is

$$e^{i\alpha} = \cos \alpha + i \sin \alpha \in S^1$$

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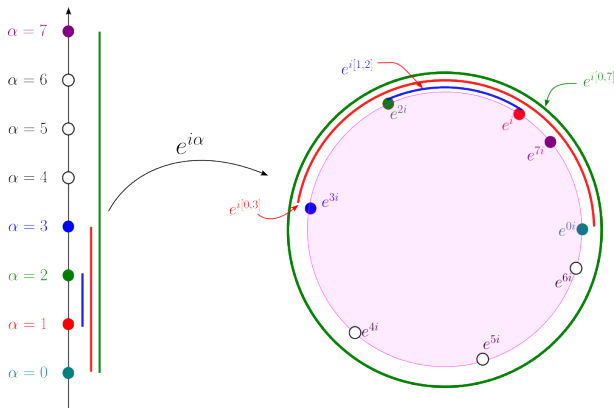


We need to extend the notion of intervals to angles.

Now S^1 has no order relation, we take a topological definition.

- An *interval* of $\mathbb{R}$ is a closed connected subset of $\mathbb{R}$.
- An *interval* of S^1 is a closed connected subset of S^1 .

An interval angle can be written as $e^{i[\alpha]}$ where $[\alpha]$ is an interval of $\mathbb{R}$.



If $w([\alpha]) < \pi$, the interval angle $e^{i[\alpha]}$ is a *minor*.

The set of minors is closed with respect to intersection.

For *majors* (non minor), the intersection is not closed. For instance,

$$e^{i[0,\pi]} \cap e^{i[-\frac{5\pi}{4},\frac{\pi}{4}]} = e^{i[-\frac{5\pi}{4},-\pi]} \cup e^{i[0,\frac{\pi}{4}]}$$

The *complementary* of an interval angle $e^{i[\alpha]}$ is the smallest interval angle $\overline{e^{i[\alpha]}}$ which contains $S^1 \setminus e^{i[\alpha]}$.

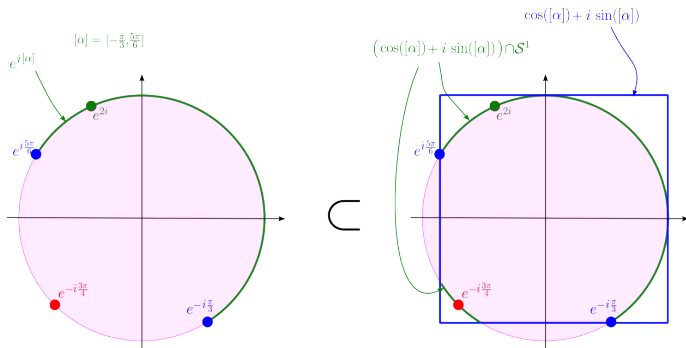
For instance

$$\overline{e^{i[0, \frac{\pi}{2}]}} = e^{i[\frac{\pi}{2}, 2\pi]}$$

Trigonometric wrapping effect. We have

$$e^{i[\alpha]} \subset (\cos([\alpha]) + i \sin([\alpha])) \cap \mathcal{S}^1$$

The equality exists for minors.

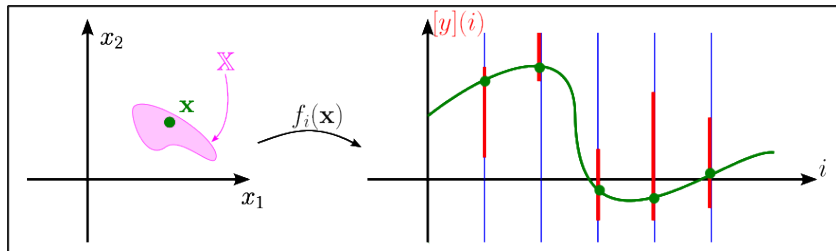


2. Set inversion

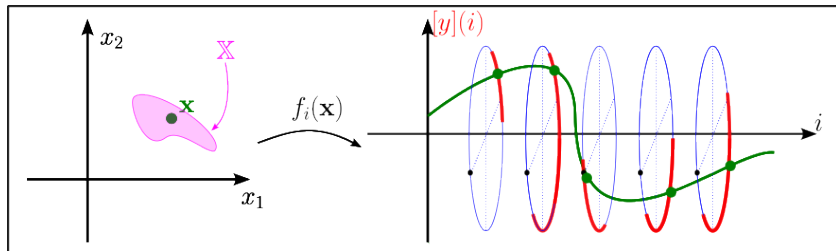
A set inversion problem:

$$\mathbb{X} = \mathbf{f}^{-1}(\mathbb{Y}).$$

In the literature, $\mathbb{Y}$ is often a box of $\mathbb{R}^m$ and $\mathbf{f}:\mathbb{R}^n \rightarrow \mathbb{R}^m$.



Y is a box



$\mathbb{Y}$ is a torus

A naive set inversion approach

Characterize the set

$$\mathbb{X} = \psi^{-1}(e^{i[\alpha]})$$

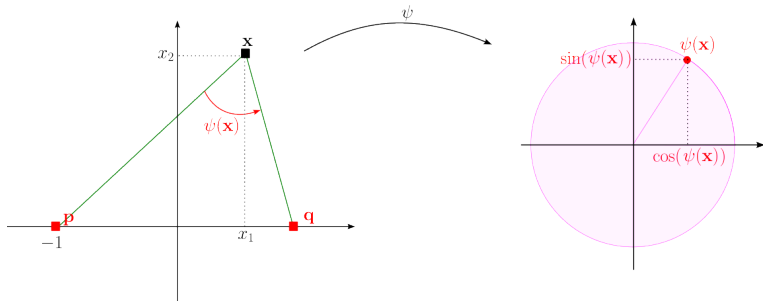
where $\psi: \mathbb{R}^n \rightarrow S^1$. Define

$$\begin{aligned}\psi_1(\mathbf{x}) &= \operatorname{Re}(\psi(\mathbf{x})) = \cos \psi(\mathbf{x}) \\ \psi_2(\mathbf{x}) &= \operatorname{Im}(\psi(\mathbf{x})) = \sin \psi(\mathbf{x})\end{aligned}$$

If $w([\alpha]) < \pi$, we have

$$\psi^{-1}(e^{i[\alpha]}) = \psi_1^{-1}(\cos([\alpha])) \cap \psi_2^{-1}(\sin([\alpha]))$$

Example: the triangle



To each point $\mathbf{x}$, we associate the angle $\psi(\mathbf{x})$

A very naive (and False) way to proceed is to write

$$\psi(\mathbf{x}) = \text{atan2}(q_2 - x_2, q_1 - x_1) - \text{atan2}(p_2 - x_2, p_1 - x_1).$$

This makes $\psi(\mathbf{x})$ discontinuous and ignore the geometry of S^1 .

We have to avoid the use of atan2.

A less naive (and safe) approach is to write:

$$\cos \psi(\mathbf{x}) = \frac{x_1^2 + x_2^2 - 1}{\sqrt{(x_1 + 1)^2 + x_2^2} \cdot \sqrt{(x_1 - 1)^2 + x_2^2}}$$

and

$$\sin \psi(\mathbf{x}) = \frac{2x_2}{\sqrt{(x_1 + 1)^2 + x_2^2} \cdot \sqrt{(x_1 - 1)^2 + x_2^2}}$$

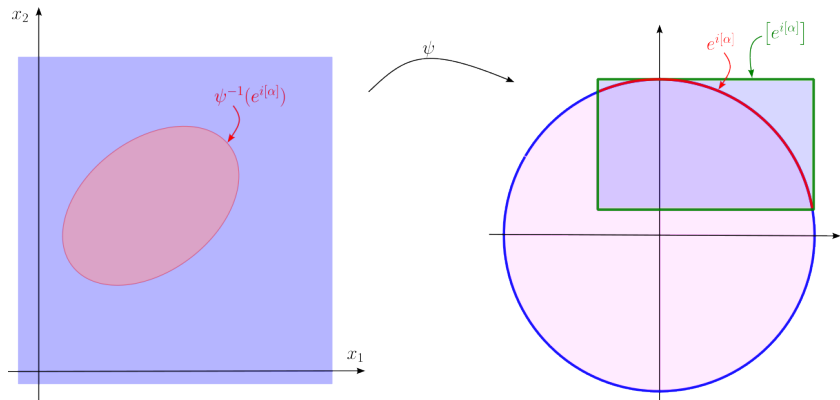
Assume that $w([\alpha]) < \pi$ and denote by $[\psi]([\mathbf{x}])$ an inclusion function for $\psi(\mathbf{x})$. We have:

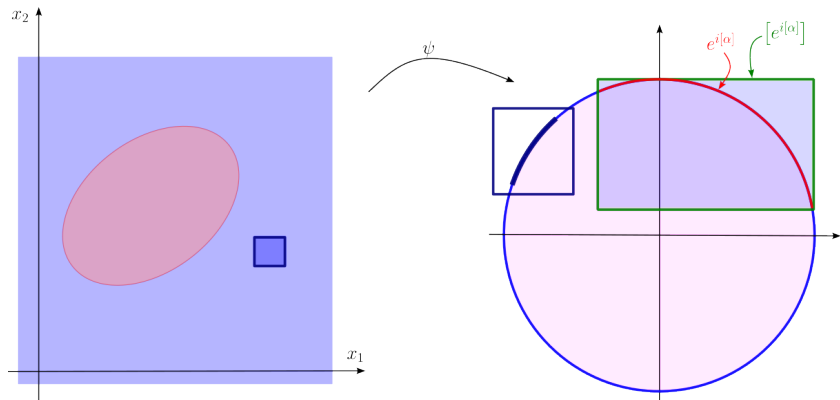
- Outer test

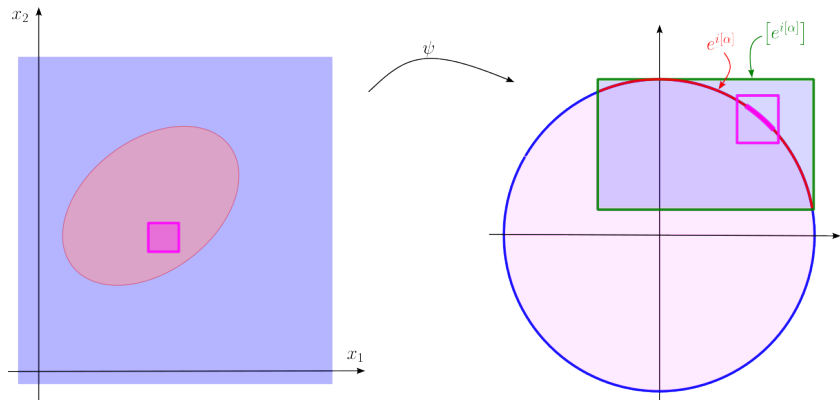
$$[\psi]([\mathbf{x}]) \cap [e^{i[\alpha]}] = \emptyset \Rightarrow [\mathbf{x}] \subset \overline{\psi^{-1}(e^{i[\alpha]})}$$

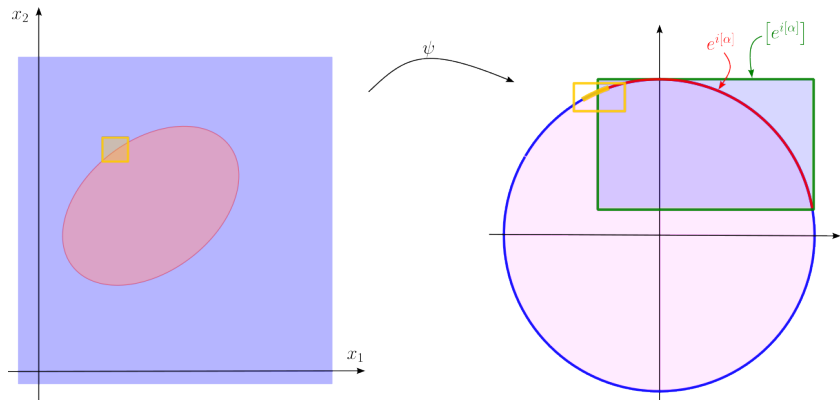
- Inner test

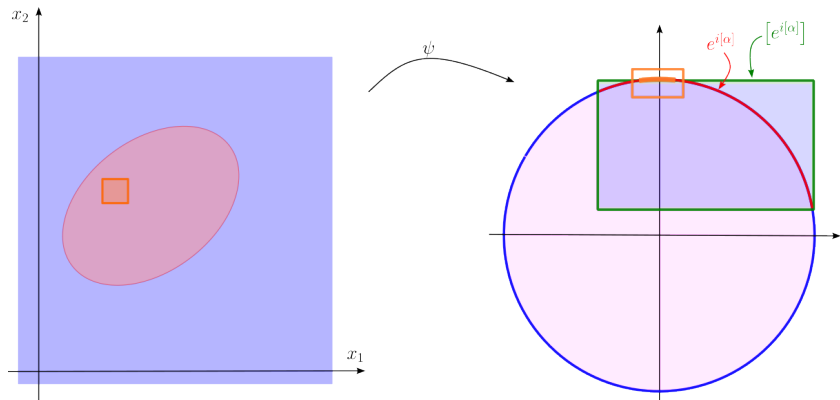
$$[\psi]([\mathbf{x}]) \subset [e^{i[\alpha]}] \Rightarrow [\mathbf{x}] \subset \psi^{-1}(e^{i[\alpha]})$$







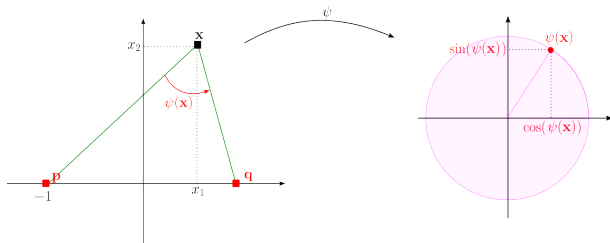




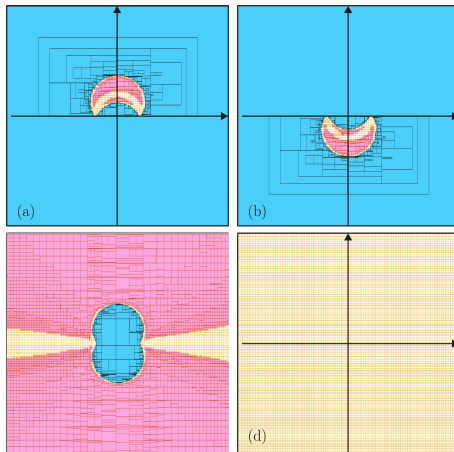
Given an interval angle $e^{i[\alpha]}$, with $w([\alpha]) < \pi$, we can use a classical set inversion algorithm to compute

$$\mathbb{X} = \psi^{-1}(e^{i[\alpha]}) = \psi^{-1} \begin{pmatrix} \cos([\alpha]) \\ \sin([\alpha]) \end{pmatrix}$$

Example: the triangle



$$\psi(x) = \frac{x_1^2 + x_2^2 - 1 + 2x_2i}{\sqrt{(x_1 + 1)^2 + x_2^2} \cdot \sqrt{(x_1 - 1)^2 + x_2^2}}$$



$$\psi^{-1}(e^{i[\alpha]}) \text{ for } [\alpha] = [1, 2], [-2, -1], [-1, 1], [1, 5]$$

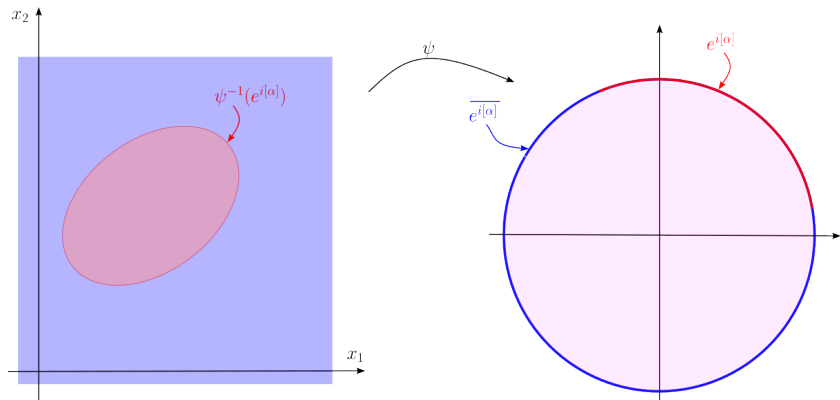
New test for interval angles

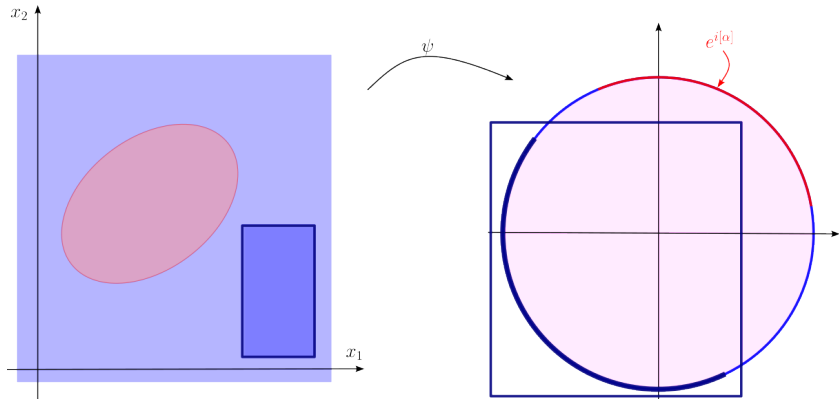
- Outer test

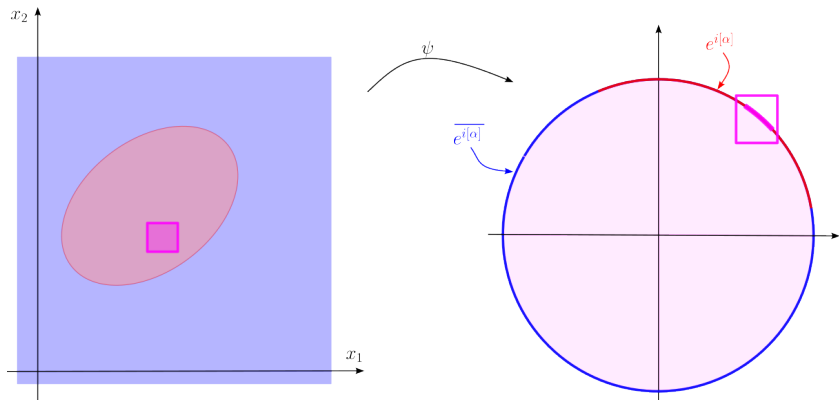
$$\psi([x]) \cap e^{i[\alpha]} = \emptyset \Rightarrow [x] \subset \overline{\psi^{-1}(e^{i[\alpha]})}$$

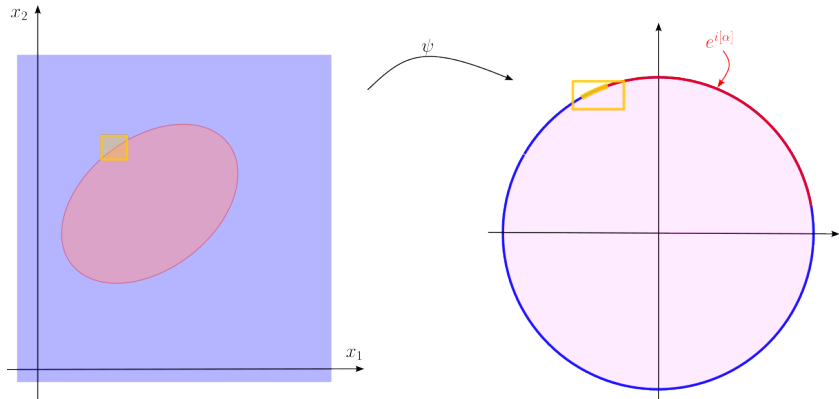
- Inner test

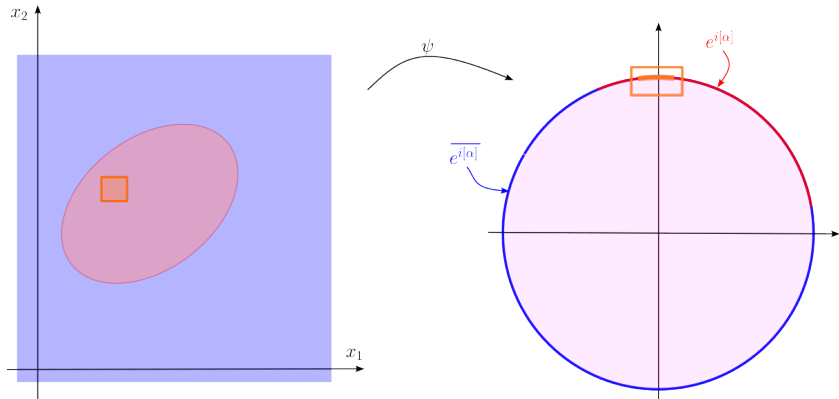
$$\psi([x]) \cap \overline{e^{i[\alpha]}} = \emptyset \Rightarrow [x] \subset \psi^{-1}(e^{i[\alpha]})$$

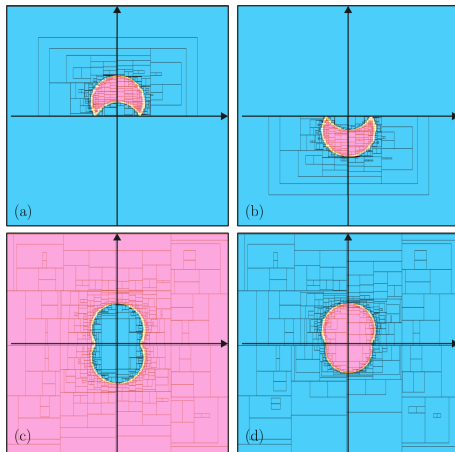












$$\psi^{-1}(e^{i[\alpha]}) \text{ for } [\alpha] = [1, 2], [-2, -1], [-1, 1], [1, 5]$$

Reduced form

We have

$$\cos^2 \psi(\mathbf{x}) + \sin^2 \psi(\mathbf{x}) = 1$$

The reduced form $\psi_r(\mathbf{x})$ for $\psi(\mathbf{x})$ is a expression (simpler) such that

$$\psi(\mathbf{x}) = \frac{\psi_r(\mathbf{x})}{\|\psi_r(\mathbf{x})\|}$$

For instance, the reduced form for

$$\psi(\mathbf{x}) = \frac{x_1^2 + x_2^2 - 1 + 2x_2i}{\sqrt{(x_1 + 1)^2 + x_2^2} \cdot \sqrt{(x_1 - 1)^2 + x_2^2}}$$

is

$$\psi_r(\mathbf{x}) = x_1^2 + x_2^2 - 1 + 2x_2i$$

Reduced test for interval angles

Define

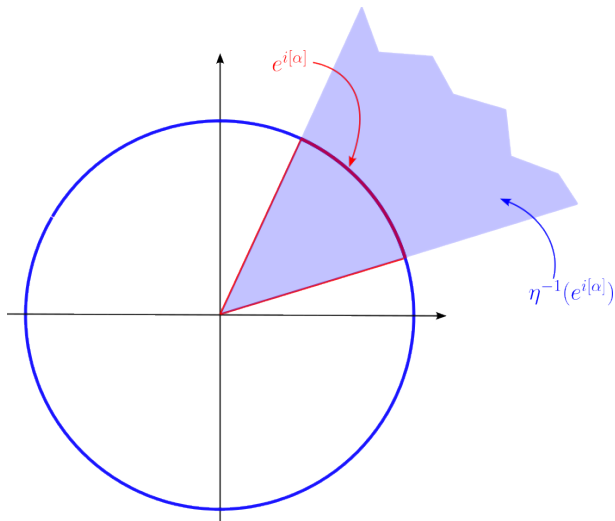
$$\eta(\mathbf{y}) = \frac{\mathbf{y}}{\|\mathbf{y}\|}$$

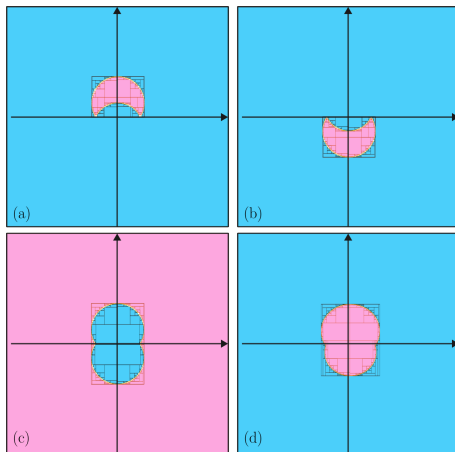
- Outer test

$$\psi_r([\mathbf{x}]) \cap \eta^{-1}(e^{i[\alpha]}) = \emptyset \Rightarrow [\mathbf{x}] \subset \overline{\psi^{-1}(e^{i[\alpha]})}$$

- Inner test

$$\psi_r([\mathbf{x}]) \cap \eta^{-1}(\overline{e^{i[\alpha]}}) = \emptyset \Rightarrow [\mathbf{x}] \subset \psi^{-1}(e^{i[\alpha]})$$



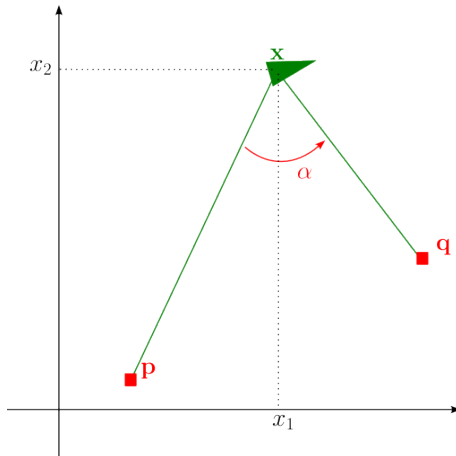


$$\psi^{-1}(e^{i[\alpha]}) \text{ for } [\alpha] = [1, 2], [-2, -1], [-1, 1], [1, 5]$$

3. Applications

Difference bearing localization

A robot at $\mathbf{x} = (x_1, x_2)$ measures the relative bearing $\alpha \in [\alpha]$ with respect to $\mathbf{p}, \mathbf{q}$.



The set of all feasible $\mathbf{x}$ is $\psi_{\mathbf{p},\mathbf{q}}^{-1}(e^{i[\alpha]})$ where

$$\psi_{\mathbf{p},\mathbf{q}}(\mathbf{x}) = \psi_0 \left(\frac{\frac{2(q_1-p_1)x_1+2(q_2-p_2)x_2+p_1^2+p_2^2-q_1^2-q_2^2}{(q_1-p_1)^2+(q_2-p_2)^2}}{\frac{2(p_2-q_2)x_1+2(q_1-p_1)x_2+2(p_1q_2-p_2q_1)}{(q_1-p_1)^2+(q_2-p_2)^2}} \right)$$

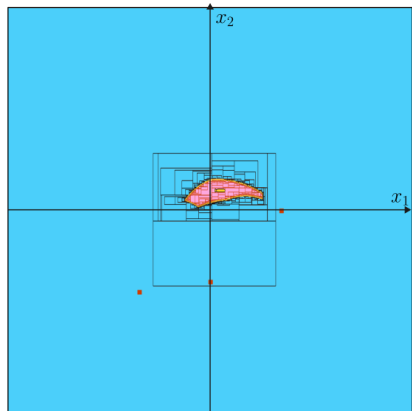
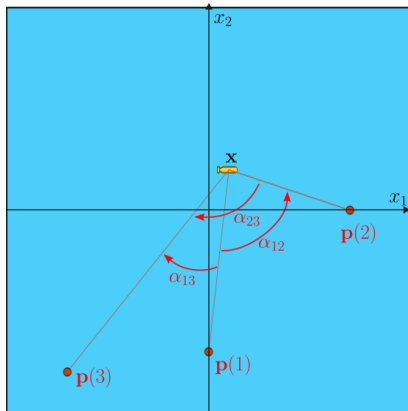
$$\psi_0(\mathbf{x}) = \frac{x_1^2+x_2^2-1+2x_2i}{\sqrt{(x_1+1)^2+x_2^2} \cdot \sqrt{(x_1-1)^2+x_2^2}}$$

Take three sensors with

$$\begin{aligned}\alpha_{12} &\in [1.16, 1.56] \\ \alpha_{32} &\in [-2.12, -1.72] \\ \alpha_{13} &\in [-0.76, -0.36]\end{aligned}$$

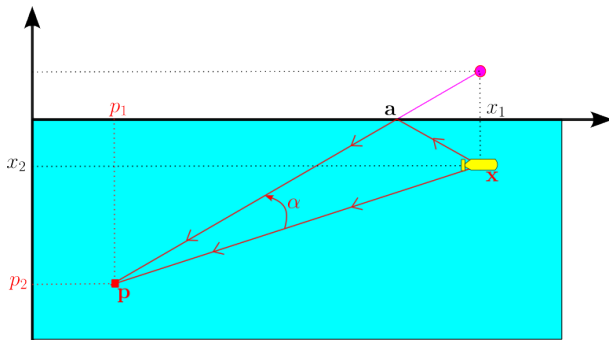
The set of all feasible position is

$$\mathbb{X} = \bigcap_{(i,j) \in \{(1,2), (1,3), (2,3)\}} \psi_{\mathbf{p}(i), \mathbf{p}(j)}^{-1}(e^{i[\alpha_{ij}]})$$



Multipath localization

A submarine, at an unknown position $\mathbf{x} = (x_1, x_2)$, emits a sound. A sensor at a known position $\mathbf{p}$ can hear this sound twice.



We have

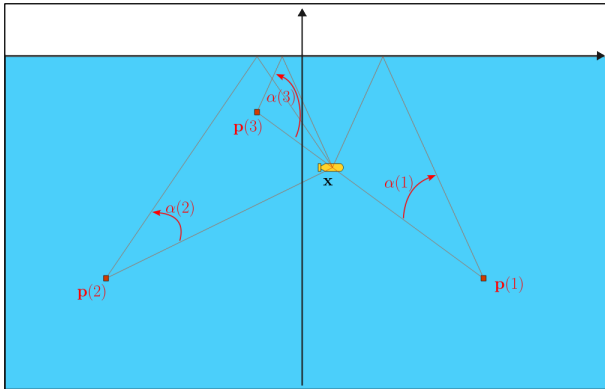
$$\psi_r(\mathbf{x}) = \begin{pmatrix} (x_1 - p_1) \left(\frac{x_2 p_1 + p_2 x_1}{p_2 + x_2} - p_1 \right) - p_2 (x_2 - p_2) \\ -p_2 (x_1 - p_1) - (x_2 - p_2) \left(\frac{x_2 p_1 + p_2 x_1}{p_2 + x_2} - p_1 \right) \end{pmatrix}.$$

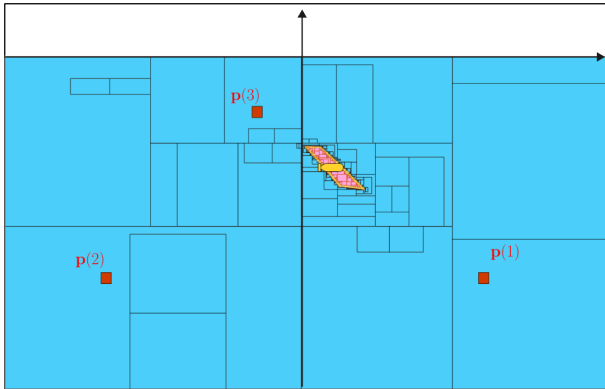
The positions of the three sensors and the interval angles are given by:

i	1	2	3
$\mathbf{p}(i)$	$\begin{pmatrix} 7 \\ -8 \end{pmatrix},$	$\begin{pmatrix} -8 \\ -8 \end{pmatrix}$	$\begin{pmatrix} -2 \\ -2 \end{pmatrix}$
$\alpha(i)$	$[-0.62, -0.42]$	$[0.41, 0.61]$	$[1.6, 1.8]$

The set of all feasible positions for the submarine is

$$\mathbb{X} = \bigcap_{k \in \{1,2,3\}} \psi_{r, \mathbf{p}(k)}^{-1}(\eta^{-1}(e^{i[\alpha](k)}))$$





References

- ① Set inversion [7] [4][5]
- ② Von Mises distribution [2][9]
- ③ Bounded-error estimation [8][11].
- ④ Interval contractors: [3]. [6]
- ⑤ Interval analysis: [10][1] [12]



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