State Estimation

Probabilistic and Bounded-error approaches

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Thursday 20 march 2009

1 Introduction

State estimation = generalization of parameter estimation.

State : set of quantity that characterize
the status of a system at a given time instant
(ex : position and speed of a ball).

State estimation
=
Estimation of the state from measurements on the system.

State may evolve with time.

A priori information assumed available

– on the way the state evolves (dynamical equation)
– on the way measurements are obtained on the system (measurement equation)

Hypotheses made

– on the measurement noise,
– state perturbations

will determine the choice for the tools used for state estimation.

2 Outline

- Discrete-time models
  - Assumptions
  - Probabilistic approach
  - Bounded-error approach
- Joint parameter and state estimation
- Continuous-time models
  - Assumptions
  - Bounded-error approach

3 Discrete-time models

Consider a system described by the discrete-time state equation

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{w}_k\right), \quad k = 0, 1, ..., \tag{1}$$

where

- $\mathbf{f}_k$ is a known function (possibly nonlinear and time-varying),
- $\mathbf{x}_k$ is the unknown state vector at time k,
- $\mathbf{w}_k$ is some unknown state perturbation vector.

Measurements satisfy

$$\mathbf{y}_\ell = \mathbf{h}_\ell(\mathbf{x}_\ell, \mathbf{v}_\ell), \quad \ell = 1, ..., k, \tag{2}$$

where

- $\mathbf{h}_\ell$ is a known function (also possibly nonlinear and time-varying),
- $\mathbf{y}_\ell$ is the measurement vector at time ℓ,
- $\mathbf{v}_\ell$ is some unknown measurement noise vector.

Depending on the nature of $\mathbf{f}_k$ and $\mathbf{h}_\ell$

and

on the information assumed available about

the state perturbation and measurement noise

various types of state estimators are available.

4 Probabilistic approach

4.1 Assumptions

– $\{\mathbf{w}_k, k \in \mathbb{N}\}$ and $\{\mathbf{v}_k, k \in \mathbb{N}\}$ are i.i.d. sequences with known pdfs,
– pdf of $\mathbf{x}_0$ based on no measurement is known.

Recursive computation of

$$p(\mathbf{x}_k|\mathbf{y}_{1:k}),$$

posterior pdf of $\mathbf{x}_k$ based on k first measurements, possible at least in principle.

Optimal solution of the state estimation problem in a Bayesian sense.

4.2 Recursive algorithm

Alternates

– *predictions*, prior pdf $p(\mathbf{x}_k|\mathbf{y}_{1:k-1})$ computed via the Chapman-Kolgomorov equation

$$p\left(\mathbf{x}_k|\mathbf{y}_{1:k-1}\right) = \int p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right) p\left(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1}\right) d\mathbf{x}_{k-1} \tag{3}$$

$\hookrightarrow$ involves the state equation

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{w}_k\right), \quad k = 0, 1, ...,$$

– *corrections,* new measurement $\mathbf{y}_k$ taken into account to update the prior pdf into the posterior pdf $p(\mathbf{x}_k|\mathbf{y}_k)$ via Bayes' rule

$$p\left(\mathbf{x}_k|\mathbf{y}_{1:k}\right) = \frac{p\left(\mathbf{y}_k|\mathbf{x}_k\right) p\left(\mathbf{x}_k|\mathbf{y}_{1:k-1}\right)}{p\left(\mathbf{y}_k|\mathbf{y}_{1:k-1}\right)}. \tag{4}$$

$\hookrightarrow$ involves the measurement equation

$$\mathbf{y}_\ell = \mathbf{h}_\ell(\mathbf{x}_\ell, \mathbf{v}_\ell), \quad \ell = 1, ..., k,$$

Usually, (3) and (4) very difficult to evaluate.

Approached solutions have to be derived.

4.3 Linear-Gaussian case

When

- $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are linear,
- the pdfs of
  - $\mathbf{x}_0$,
  - the state perturbations
  - the measurement noise

  are Gaussian with known mean and covariance matrix

Kalman filter is the optimal solution (Sorenson, 1985).

4.4 Non-linear-Gaussian case

When

- $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are non-linear,
- the pdfs of
  - $\mathbf{x}_0$,
  - the state perturbations
  - the measurement noise

  are Gaussian with known mean and covariance matrix

Extended Kalman filter (Gelb, 1974; Anderson and Moore, 1979)

$\hookrightarrow$ linearization of the state and measurement equations

$\hookrightarrow$ Gaussian approximation of all pdfs

$\oplus$ Simple implementation

$\ominus$ Actual state may get lost

Unscented Kalman filter (Julier and Uhlmann, 2004)

$\hookrightarrow$ linearization of the state and measurement equations

$\hookrightarrow$ approximation of the *a priori* pdf by some well-selected points

$\oplus$ Simple implementation

$\oplus$ Performs better than the Extended Kalman filter

$\ominus$ Actual state may still get lost

Grid-based approach (Terwiesch and Agarwal, 1995; Burgard *et al.*, 1996)

$\hookrightarrow$ discretisation of the state-space using a fixed grid

$\oplus$ Non-linear treatment

$\oplus$ Integrals replaced by discrete sums more easily evaluated

$\odot$ Accuracy depends on the size of the grid

$\ominus$ Complexity depends on the size of the grid and dimension of the state

$\ominus$ Actual state may still get lost

Particle filtering approach (Gordon *et al.*, 1993; Kitagawa, 1996; Pitt and Shephard, 1999; Arulampalam *et al.*, 2002)

$\hookrightarrow$ approximation of the pdfs using clouds of points

$\oplus$ Non-linear treatment

$\oplus$ Integrals replaced by discrete sums more easily evaluated

$\odot$ Accuracy depends on the number of points

$\ominus$ Rather complex management of the cloud

$\ominus$ Actual state may still get lost

5 Bounded-error approach

5.1 Assumptions

Supports with known shapes are available for $\mathbf{w}_k, \mathbf{v}_k$ and $\mathbf{x}_0$

- $\mathbf{w}_k \in \mathbb{W}_k$, $k \in \mathbb{N}$ with known $\{\mathbb{W}_k, k \in \mathbb{N}\}$,
- $\mathbf{v}_k \in \mathbb{V}_k$, $k \in \mathbb{N}$ with known $\{\mathbb{V}_k, k \in \mathbb{N}\}$,
- $\mathbf{x}_0 \in \mathbb{X}_0$, known.

Recursive computation of the set

$$\mathbb{X}_{k|1:k}$$

of all state vectors that are compatible with

- the supports,
- the measurements,
- the models.

5.2 Recursive algorithm

Alternates

– *predictions*, set $\mathbb{X}_{k|1:k}$ assumed available, predicted set

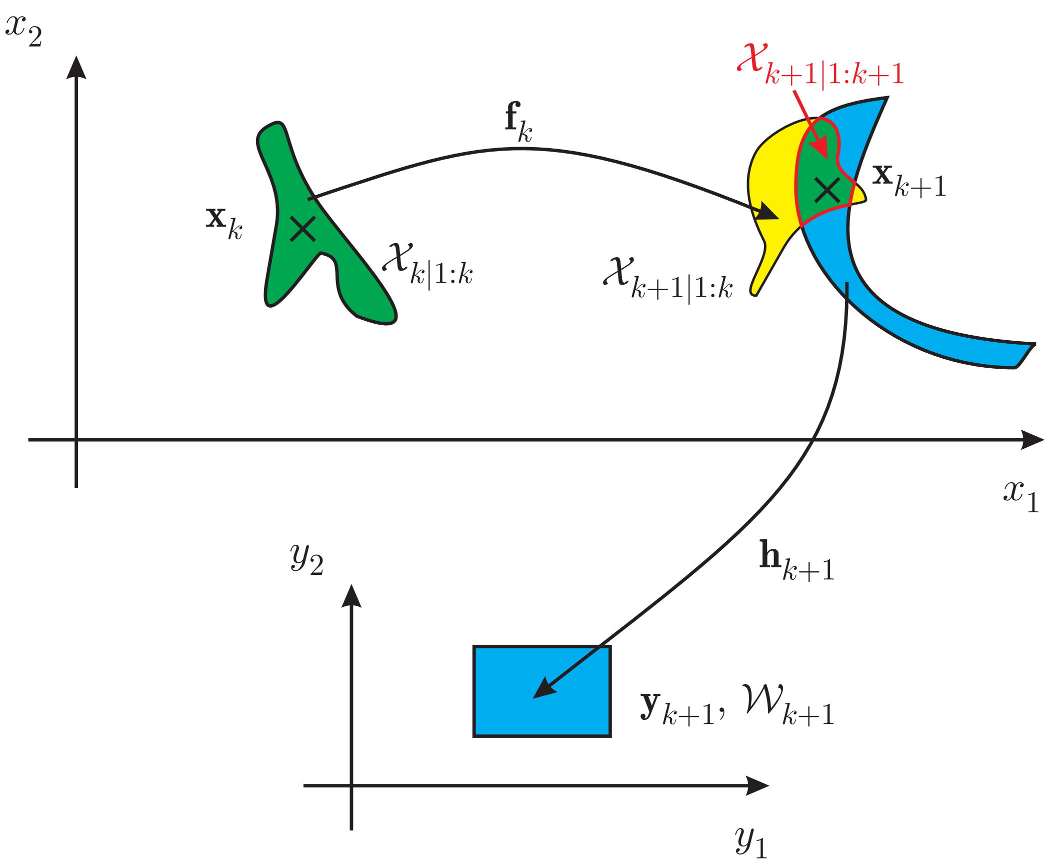
$$\mathbb{X}_{k+1|1:k} = \mathbf{f}_k \left(\mathbb{X}_{k|1:k}, \mathbb{V}_k\right) \tag{5}$$

$\hookrightarrow$ involves the state equation.

– *corrections*, new measurement $\mathbf{y}_{k+1}$ taken into account to update $\mathbb{X}_{k+1|1:k}$, corrected set

$$\mathbb{X}_{k+1|1:k+1} = \left\{\mathbf{x} \in \mathbb{X}_{k+1|1:k} : \mathbf{y}_{k+1} \in \mathbf{h}_{k+1}(\mathbb{X}_{k+1|1:k}, \mathbb{W}_{k+1})\right\}. \tag{6}$$

$\hookrightarrow$ involves the measurement equation.



When the hypotheses about the support are not violated

$\Downarrow$

Guaranteed state estimator

(no compatible state vector may get lost)

BUT

Implementation very difficult in general

$\Downarrow$

Approximate characterization of $\mathbb{X}_{k|1:k}$ using boxes, ellipsoids...

Algorithms still guaranteed, provided that approximation $\widehat{\mathbb{X}}_{k|1:k}$ of $\mathbb{X}_{k|1:k}$ satisfies

$$\mathbb{X}_{k|1:k} \subset \widehat{\mathbb{X}}_{k|1:k}, \;\; k = 1, \ldots$$

5.3 Linear case

When

- $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are linear,
- state perturbation and measurement noise are
  - additive,
  - with elliposidal support

Ellipsoidal approximation for $\widehat{\mathbb{X}}_{k|1:k}$,
see *e.g.*, (Schweppe, 1968; Bertsekas and Rhodes, 1971; Schweppe, 1973) and (Durieu *et al.*, 1996; Durieu *et al.*, 2001).

Weaker hypotheses on $\mathbf{f}_k$ (perturbations) may be considered (Chernousko and Rokityanskii, 2000; Calafiore and El Ghaoui, 2004; Polyak *et al.*, 2004).

More details in the talk by S. Lesecq.

When

- $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are linear,
- state perturbation and measurement noise are
  - additive,
  - with boxes as support

Parallelotopic approximation for $\widehat{\mathbb{X}}_{k|1:k}$,
see *e.g.*, (Chisci *et al.*, 1996).

Exact polytopic descriptionfor $\mathbb{X}_{k|1:k}$,
(Shamma and Tu, 1999).

5.4 Non-linear case

When $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are non-linear, $\mathbb{X}_{k|1:k}$ may be

- non-convex
- non-connected.

When

- $\mathbf{h}_\ell$ is linear
- State perturbation and measurement noise are
  - additive,
  - with boxes as support

Parallelotopic approximation for $\widehat{\mathbb{X}}_{k|1:k}$,
(Shamma and Tu, 1997).

When

- $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are non-linear,
- state perturbation and measurement noise are
  - with boxes or subpavings as support

Subpaving approximation for $\widehat{\mathbb{X}}_{k|1:k}$,
(Kieffer *et al.*, 1998; Kieffer *et al.*, 2002).

6 Joint parameter and state estimation

Assume that

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{p}_k, \mathbf{w}_k\right), \quad k = 0, 1, ..., \tag{7}$$

and

$$\mathbf{y}_\ell = \mathbf{h}_\ell(\mathbf{x}_\ell, \mathbf{p}_k, \mathbf{v}_\ell), \quad \ell = 1, ..., k, \tag{8}$$

where

- $\mathbf{f}_k$ and $\mathbf{h}_\ell$ are known functions (possibly nonlinear and time-varying),
- $\mathbf{x}_k$ is the unknown state vector at time k,
- $\mathbf{p}_k$ is some unknown parameter vector
- $\mathbf{w}_k$ and $\mathbf{v}_\ell$ are some unknown state perturbation and measurement vectors.

Joint estimation of $\mathbf{x}_k$ and $\mathbf{p}_k$ possible
by defining and extended state vector

$$\mathbf{x}_k^{\mathrm{e}} = \left(\mathbf{x}_k^{\mathrm{T}}, \mathbf{p}_k^{\mathrm{T}}\right)^{\mathrm{T}}$$

Assumptions are required for the variations of $\mathbf{p}_k$ with time

– constant

$$\mathbf{p}'_k = 0$$

and the following extended state equation may be defined

$$\mathbf{x}^{\mathrm{e}}_{k+1} = \begin{pmatrix} \mathbf{f}_k\left(\mathbf{x}^{\mathrm{e}}_k, \mathbf{w}_k\right) \\ \mathbf{0} \end{pmatrix} \qquad k = 0, 1, ...,$$

– slowly varying

$$\mathbf{p}'_k = \mathbf{w}^{\mathrm{p}}_\ell$$

and the following extended state equation may be defined

$$\mathbf{x}^{\mathrm{e}}_{k+1} = \begin{pmatrix} \mathbf{f}_k\left(\mathbf{x}^{\mathrm{e}}_k, \mathbf{w}_k\right) \\ \mathbf{w}^{\mathrm{p}}_\ell \end{pmatrix} \qquad k = 0, 1, ...,$$

– ...

7 Continuous-time models

Assume now that the dynamical equation is continuous-time

$$\mathbf{x}' = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}\left(\mathbf{x}\left(t\right), \mathbf{v}\left(t\right), t\right), \tag{9}$$

and that discrete-time measurements are available

$$\mathbf{y}\left(t_k\right) = \mathbf{h}\left(\mathbf{x}\left(t_k\right), \mathbf{w}\left(t_k\right), t_k\right). \tag{10}$$

– Continuous-time extensions of the Kalman filter
– Algebraic estimation techniques

7.1 Bounded-error context

When **f** and **h** are linear,

Ellipsoidal bounding still possible,
see (Schweppe, 1968) and (Bertsekas and Rhodes, 1971).

When **f** and **h** are non-linear,

Box approximation,
see the interval observer
(Alcaraz-González *et al.*, 1999; Gouzé *et al.*, 2000; Rapaport and Gouzé, 2003) and (Moisan *et al.*, 2009)

(Raissi *et al.*, 2004; Meslem *et al.*, 2008)

Subpaving approximation
(Jaulin, 2002; Kieffer and Walter, 2005; Kieffer and Walter, 2006).

Most of these techniques require
for the prediction step,
guaranteed numerical integration of the state equation,
see, *e.g.*, (Moore, 1966; Berz and Makino, 1998; Nedialkov and Jackson, 2001).

Correction step implemented as in the discrete-time case.

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Bounded-error State Estimation

Interval approach

—

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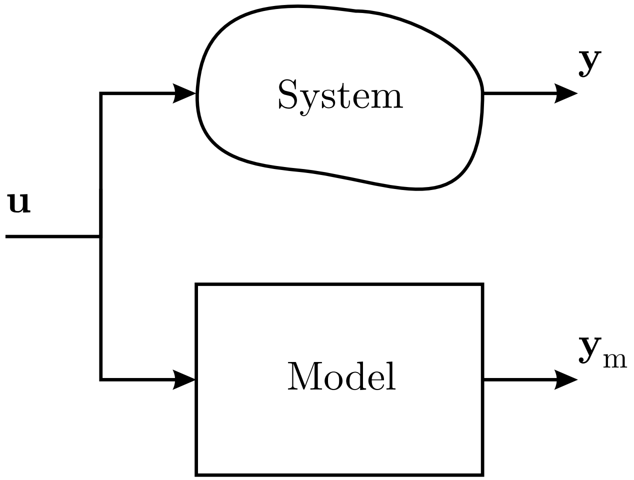
March 18, 2009

Content

- Bounded-error state estimation using interval analysis
  - Discrete-time
  - Continuous-time

1 Discrete-time state estimation

1.1 Introduction



Discrete-time state equation :

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k, \mathbf{w}_k, \mathbf{u}_k\right).$$

Observation equation :

$$\mathbf{y}_k = \mathbf{h}_{\mathbf{k}}\left(\mathbf{x}_k\right) + \mathbf{v}_k, \quad k = 1, \ldots, N.$$

Problem :

Evaluate state $\mathbf{x}_k$ using all available information.

1.2 Recursive nonlinear state estimation in a bounded-error context

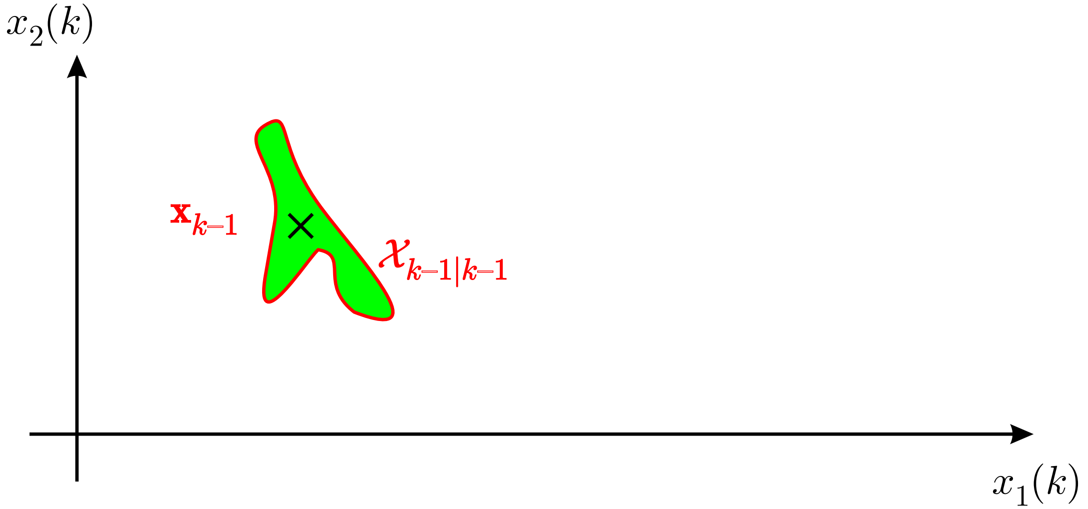
Hypotheses :

- at $k = 0$, $\mathbf{x}_0 \in [\mathbf{x}_0]$,
- $\mathbf{w}_k \in [\underline{\mathbf{w}}_k, \overline{\mathbf{w}}_k]$, known for all past k,
- $\mathbf{v}_k \in [\underline{\mathbf{v}}_k, \overline{\mathbf{v}}_k]$, known for all past k,
- $\mathbf{u}_k$, known for all past k.

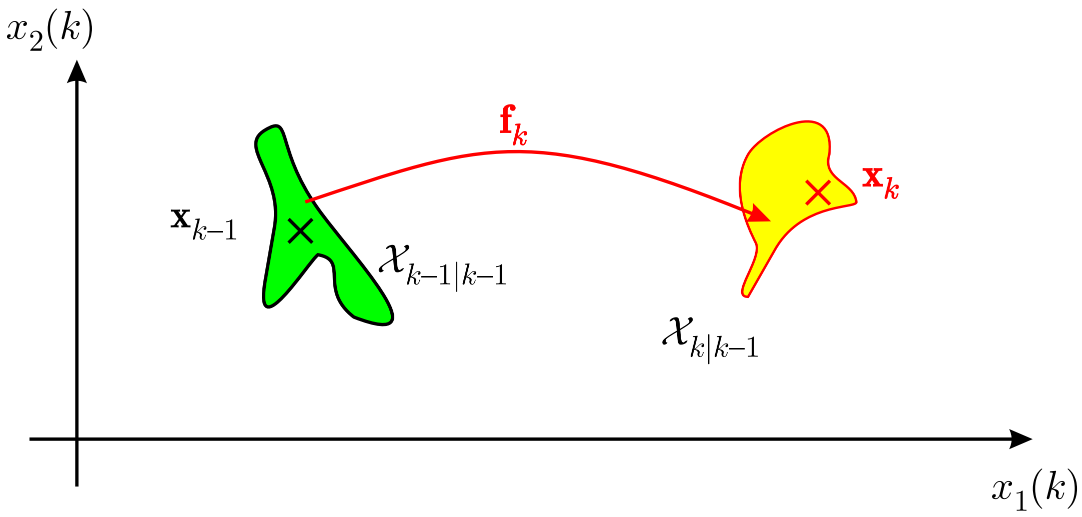
Problem :

Characterize set $\mathcal{X}_{k|k}$ of all $\mathbf{x}_k$ compatible with hypotheses.

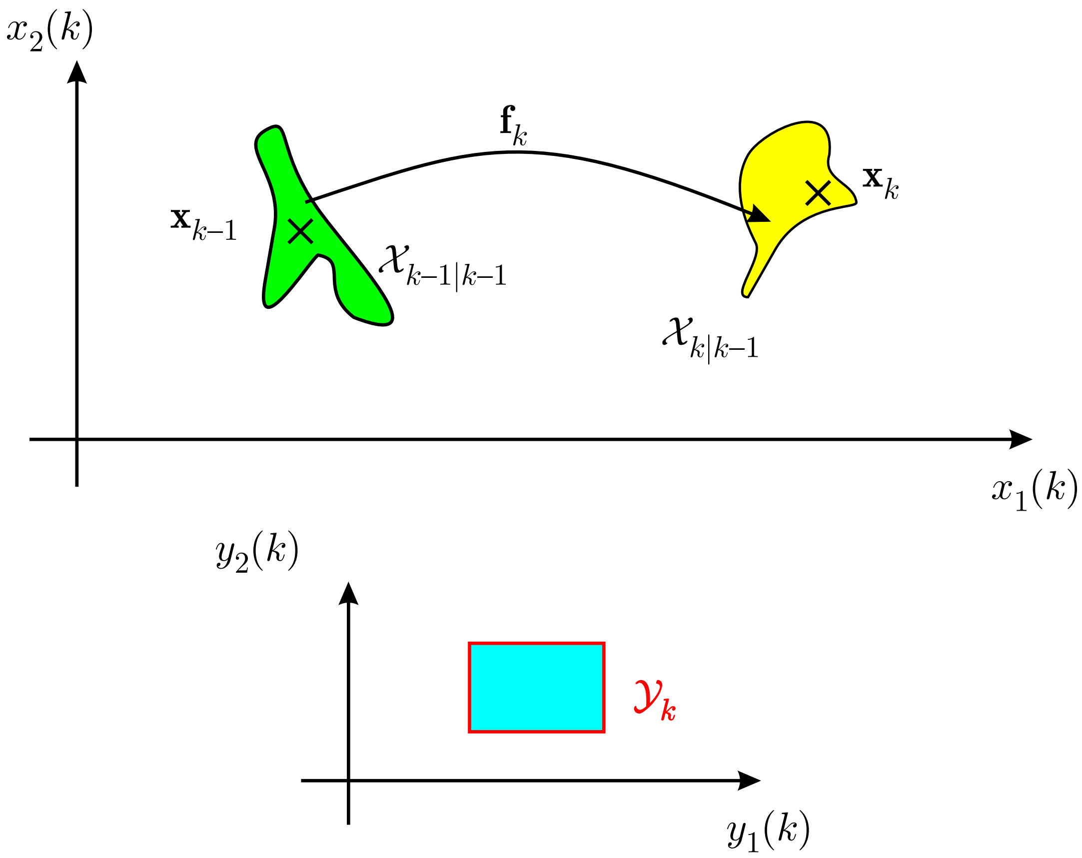
1.3 Idealized recursive state estimation



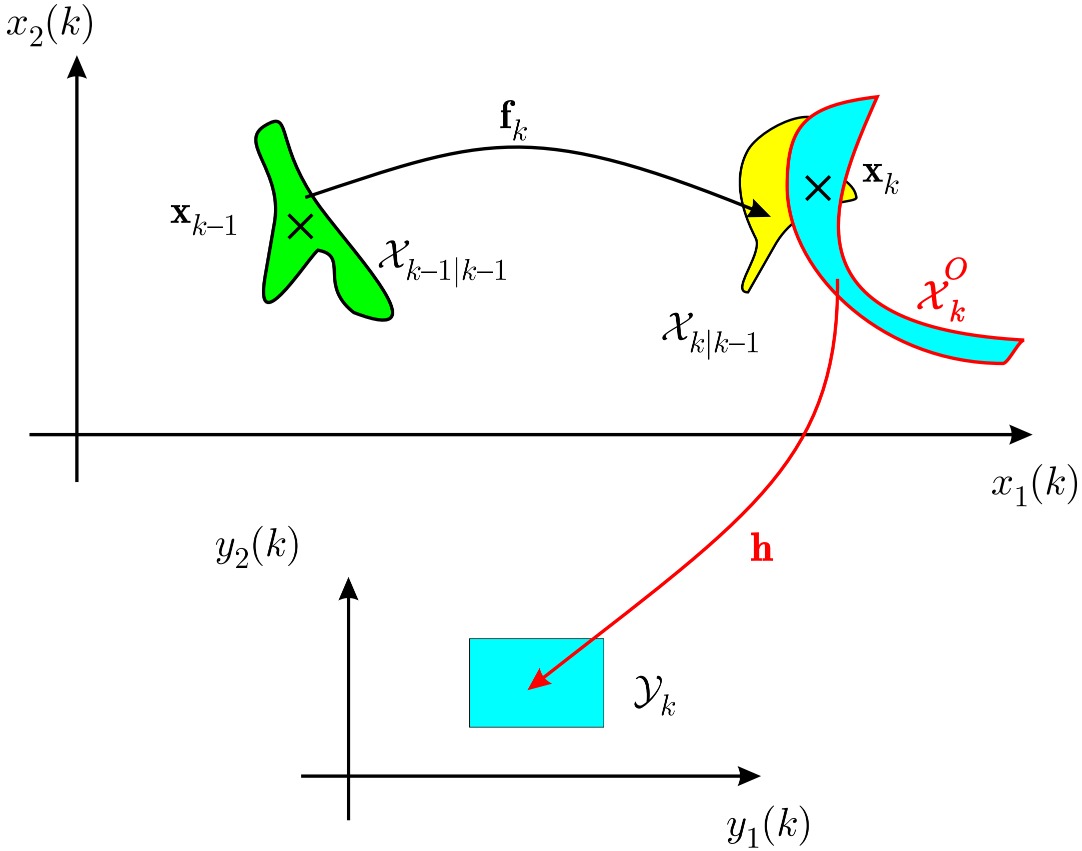
Idealized recursive state estimation



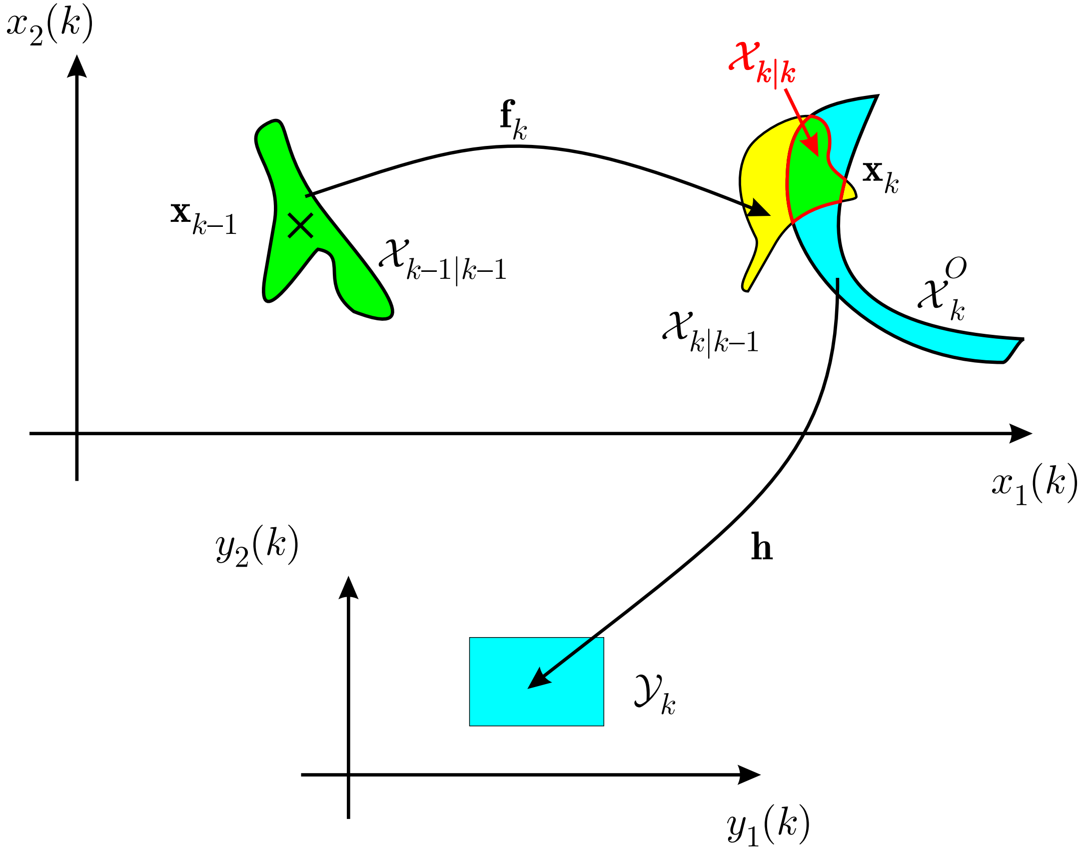
Idealized recursive state estimation



Idealized recursive state estimation

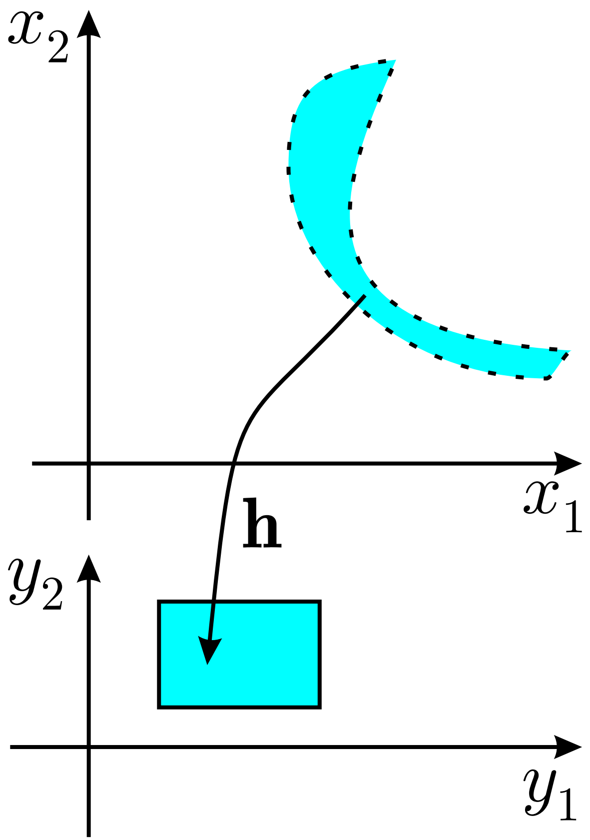


Idealized recursive state estimation



Prediction and correction steps alternate

1.4 Correction step



Set-inversion problem :

Find

$$\mathcal{X}^{O}_{k+1|k+1} = \mathbf{h}^{-1}_{k+1}\left(\mathbf{y}_{k+1} - [\mathbf{v}_{k+1}]\right).$$

Solution provided by SIVIA.

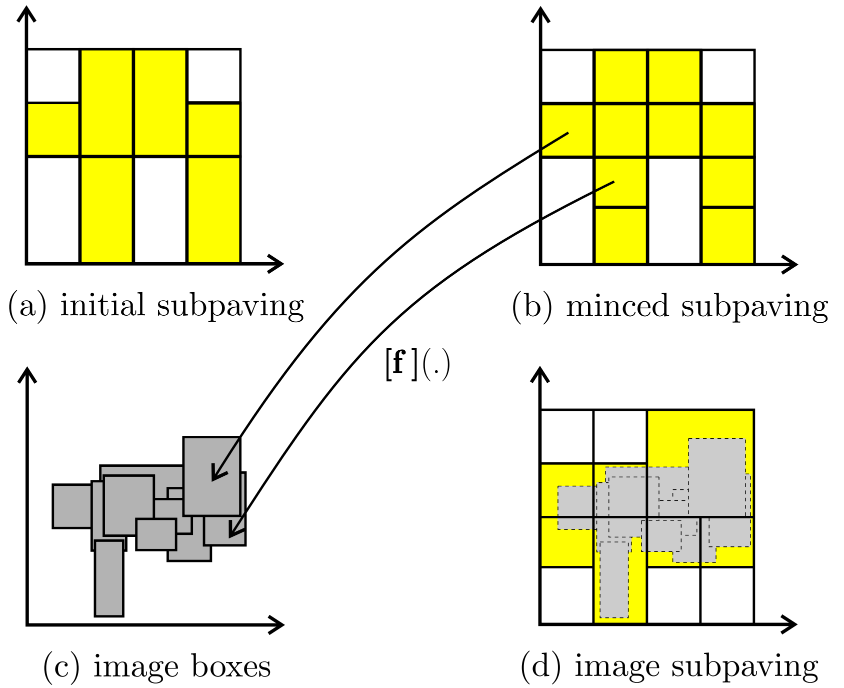
(similar to parameter estimation problem with one measurement)

1.5 Prediction step

With discrete-time state equation

$$\mathbf{x}_{k+1} = \mathbf{f}_k \left(\mathbf{x}_k, \mathbf{w}_k, \mathbf{u}_k \right).$$

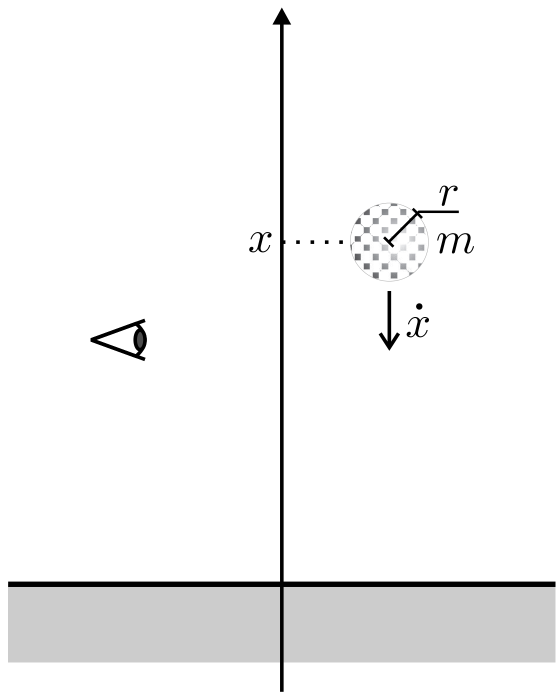
$\Longrightarrow$ ImageSp



(a) initial subpaving (b) minced subpaving (c) image boxes (d) image subpaving

1.6 Simple example: bouncing ball

Ball bouncing on floor, mass m, radius r



State

$$\mathbf{x} = (x, \dot{x})^{\mathrm{T}}$$

Sampling period $T = 0.1$ s

No friction, rigid ball

Observation:

$$y_{k+1} = (1\ 0)\,\mathbf{x}_k + [-0.2, 0.2]$$

Prediction:

$$\mathbf{x}_{k+1} = \mathbf{f}_k\left(\mathbf{x}_k\right)$$

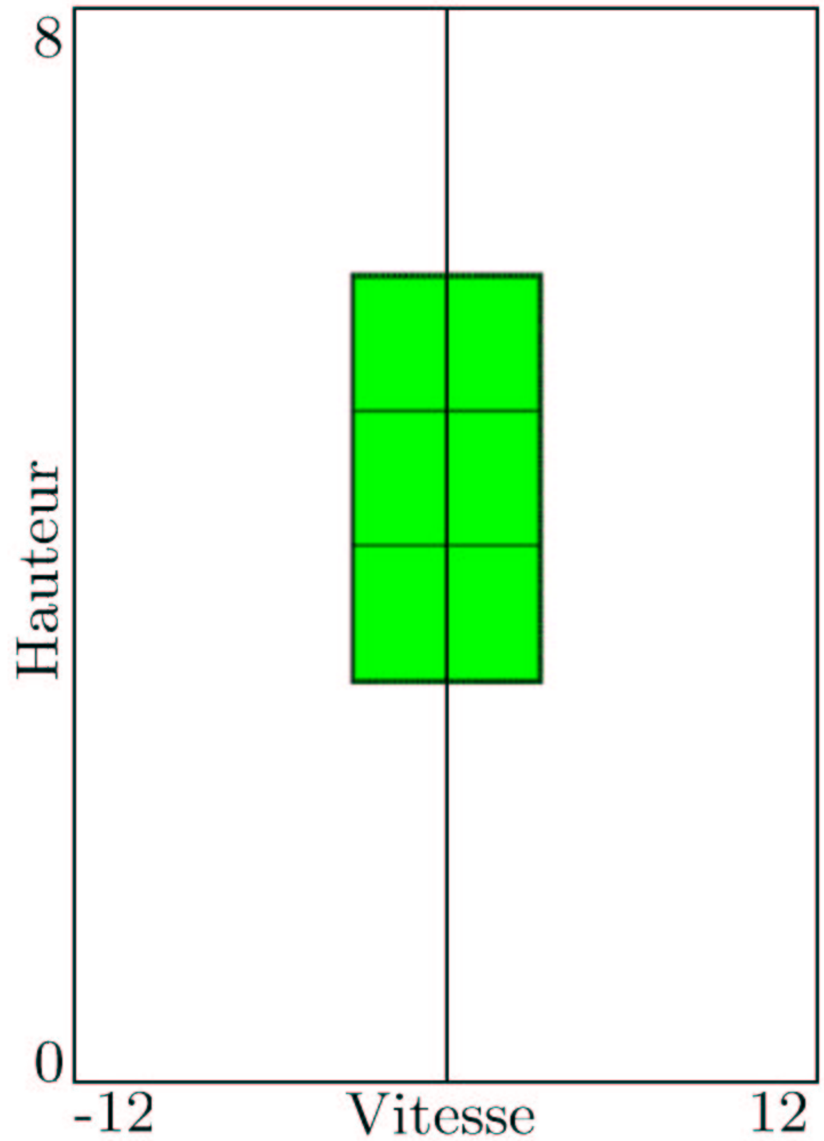
obtained by exact discretisation.

Nonlinear equations due to the bounce

$$\dot{x} \longrightarrow -\dot{x}$$

Actual initial state (unknown)

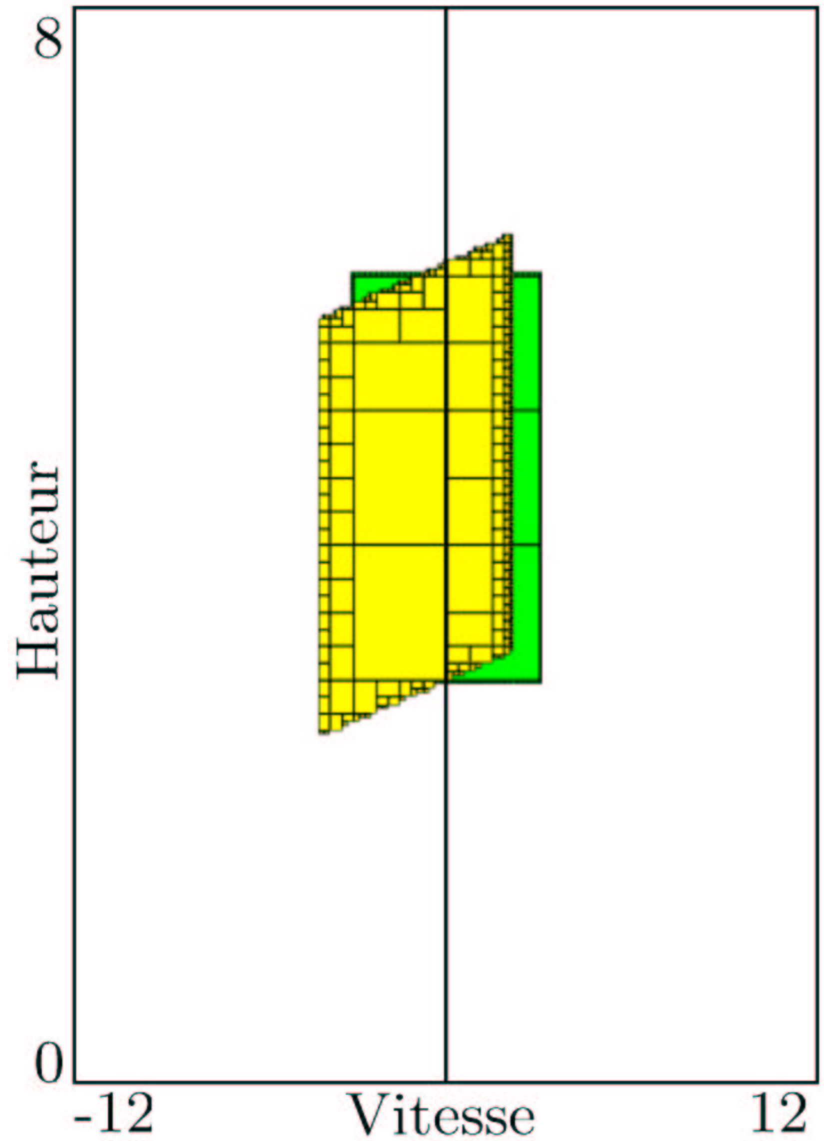
$$\mathbf{x}_0 = 5.2 \text{ m},0 \text{ m.s}^{-1}$$



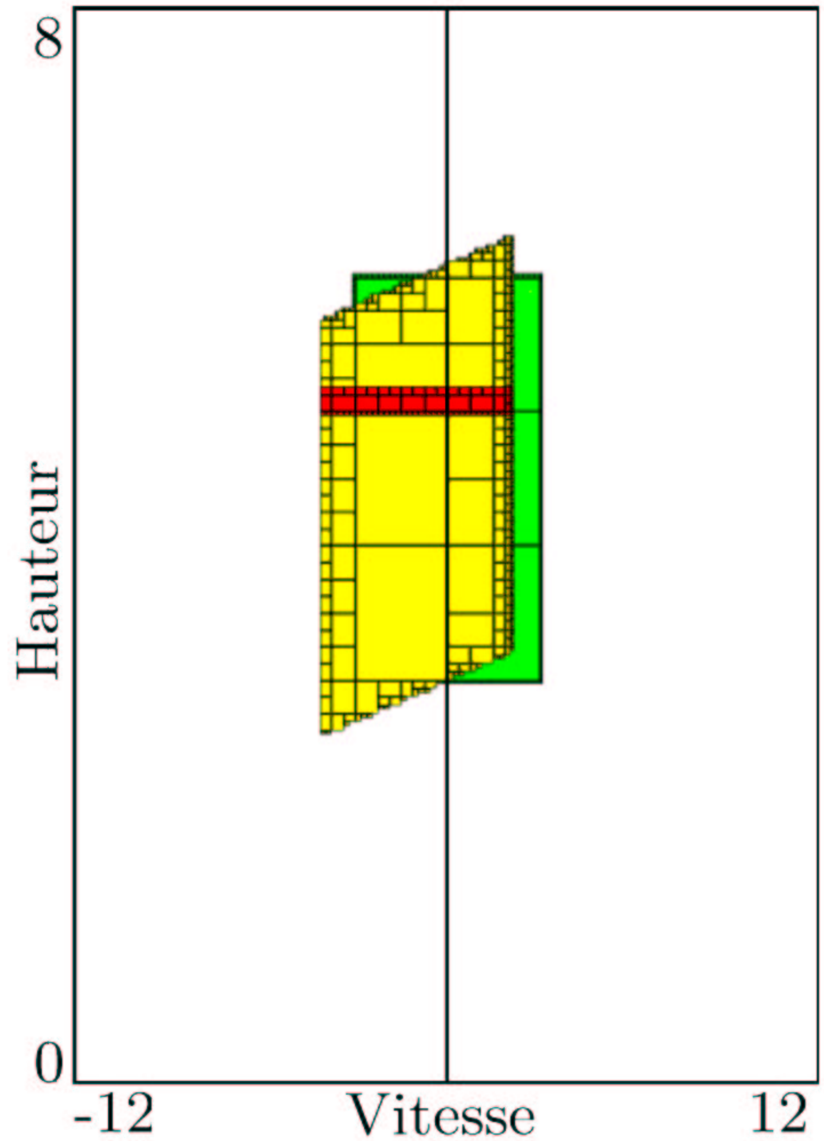
Initial search box

$$[\mathbf{x}_0] = [3,6] \times [-3,3]$$

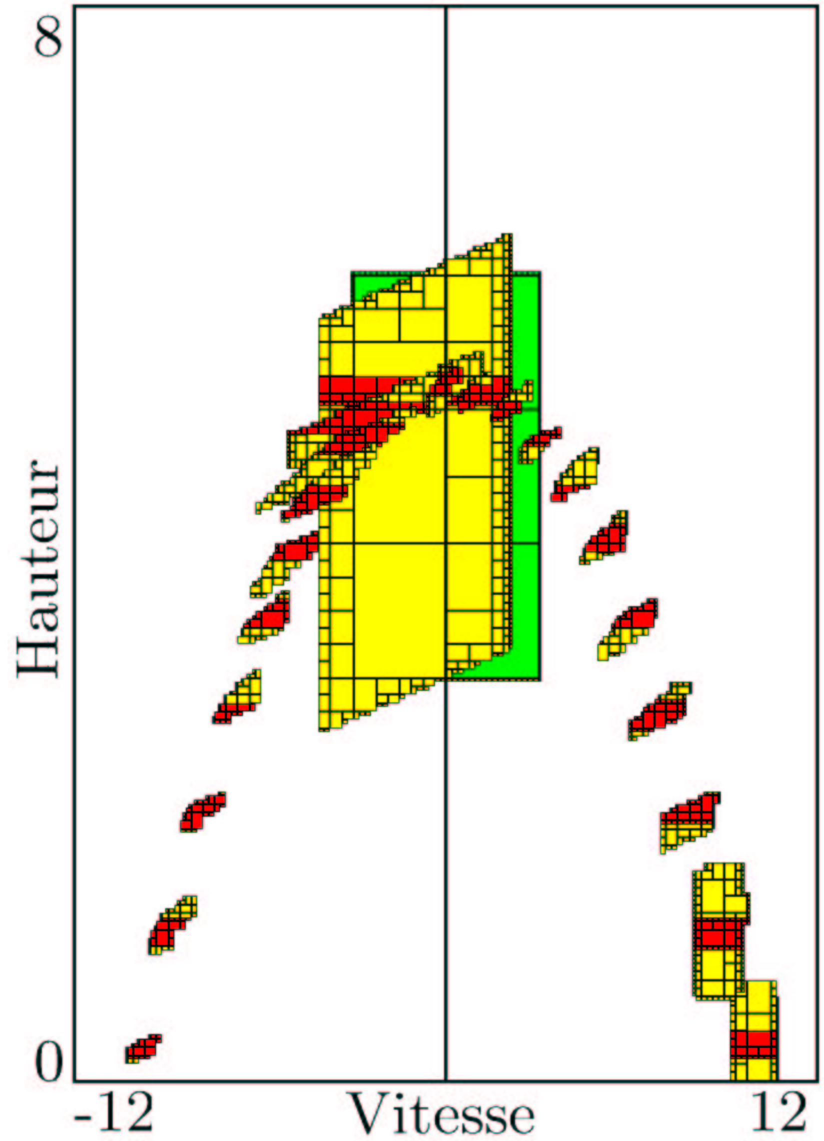
$k = 1$, prediction



$k = 1$, correction



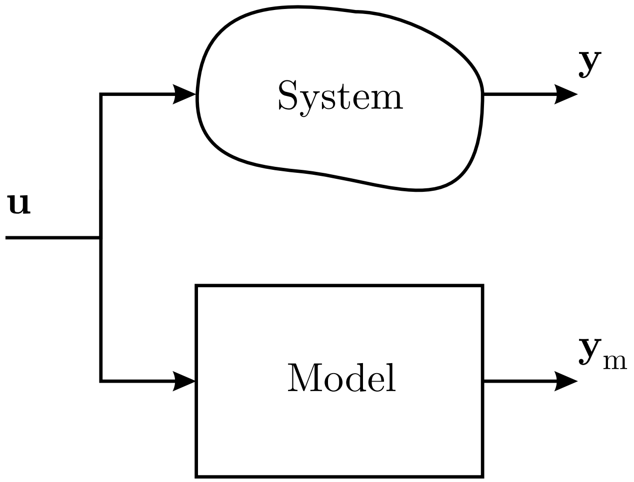
$k = 1 \dots 20$,



Takes 0.2 s on a AMD K6 at 1.5 GHz

2 Continuous-time state estimation

2.1 Introduction



Continuous-time state equation :

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{w}, \mathbf{u}\right).$$

Observation equation :

$$\mathbf{y}\left(t_i\right) = \mathbf{h}\left(\mathbf{x}\left(t_i\right)\right) + \mathbf{v}\left(t_i\right), \quad i = 1, \ldots, N.$$

Problem :

Evaluate state $\mathbf{x}$ using all available information.

2.2 Recursive nonlinear state estimation in a bounded-error context

Hypotheses :

- at t_0, $\mathbf{x}(t_0) \in [\mathbf{x}_0]$,
- $\mathbf{w}(t) \in [\underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t)]$, known for all past t,
- $\mathbf{v}(t_i) \in [\underline{\mathbf{v}}(t_i), \overline{\mathbf{v}}(t_i)]$, known for all past t_i,
- $\mathbf{u}(t)$, known for all past t.

Problem :

Characterize set $\mathcal{X}(t)$ of all $\mathbf{x}(t)$ compatible with hypotheses.

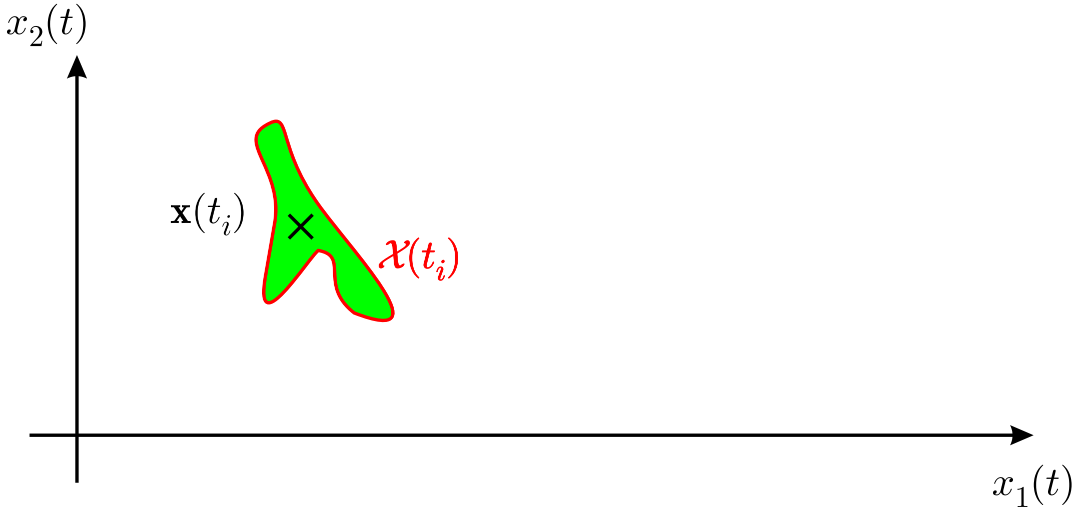
Previous results

- (Kieffer *et al*, CDC, 98) :
  - discrete-time,
  - set description with subpavings.
- (Gouzé *et al*, J. Ecol. Mod., 00) :
  - continuous-time,
  - uncertain state equation,
  - cooperative uncertain state equation bounded between cooperative systems,
  - set description with boxes.
- (Jaulin, Automatica, 02) :
  - continuous-time,
  - no state perturbations,
  - guaranteed numerical integration of non-punctual boxes,
  - set description with subpavings.
- (Raissi *et al.*, Automatica, 04)
  - continuous-time,
  - no state perturbations,
  - improved guaranteed numerical integration of non-punctual boxes,
  - set description with boxes.

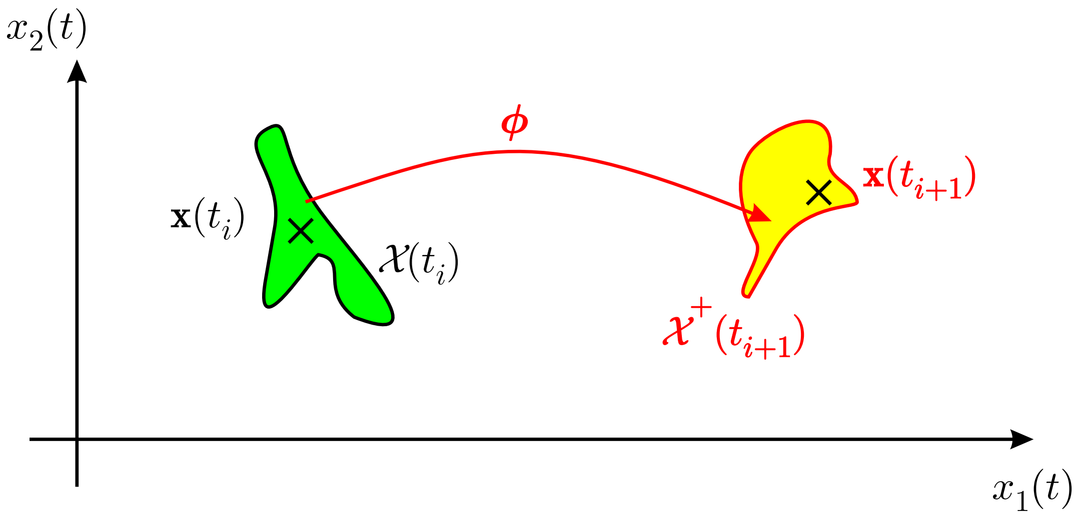
Present context

- continuous-time,
- state perturbations,
- uncertain state equations bounded between point dynamical systems,
- set description with subpavings,

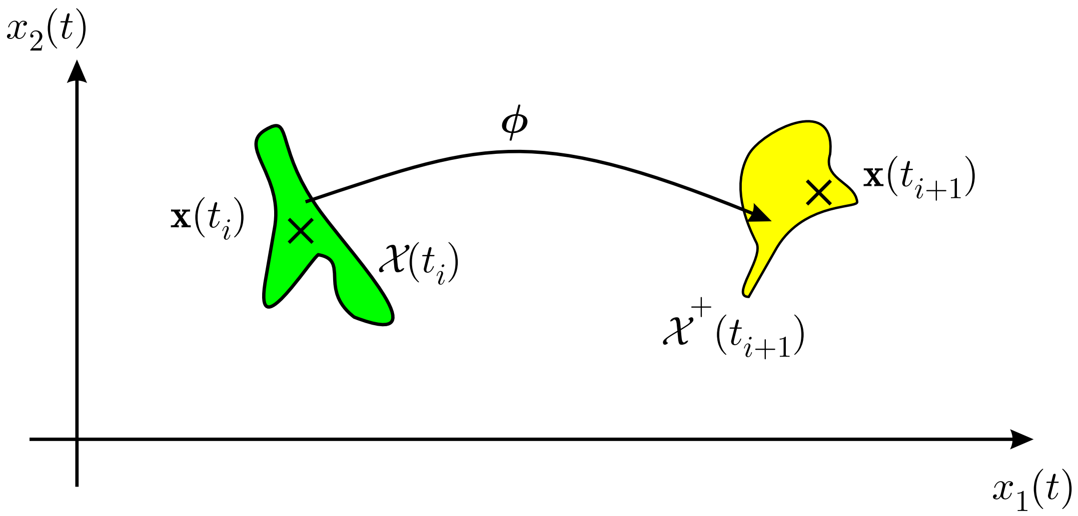
2.3 Idealized recursive state estimation

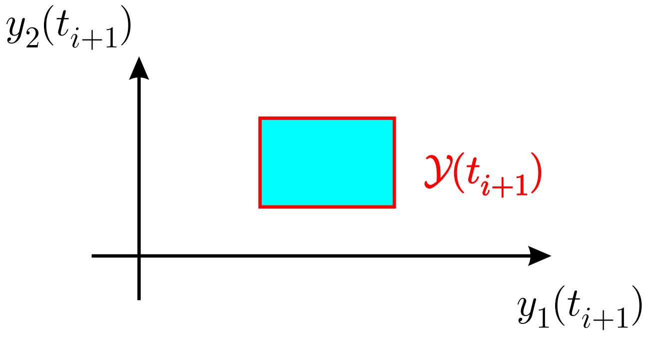


Idealized recursive state estimation

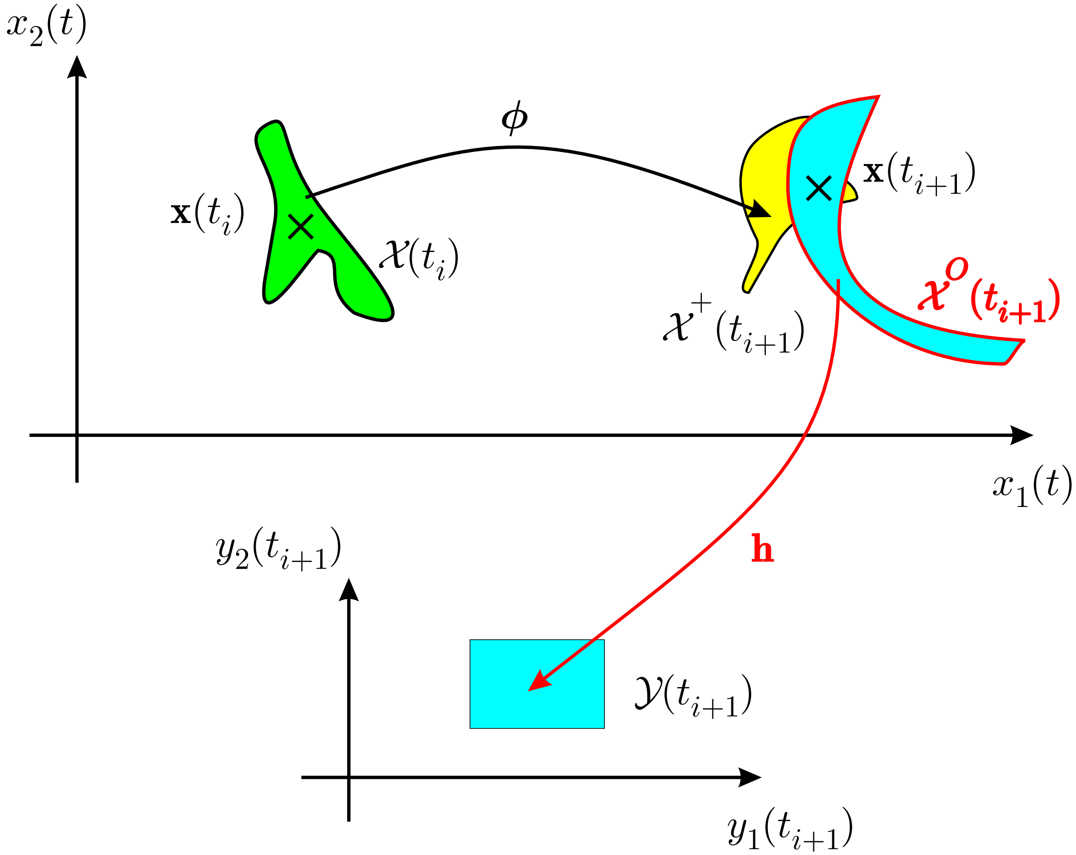


Idealized recursive state estimation

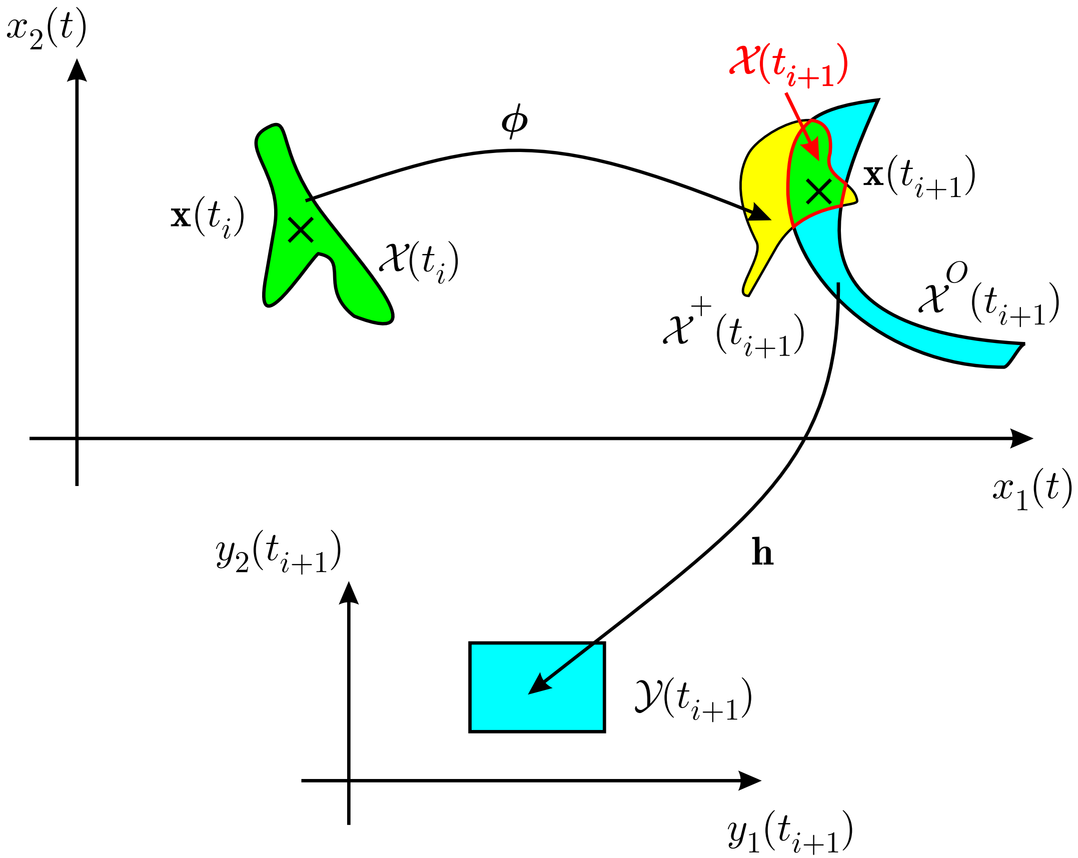




Idealized recursive state estimation

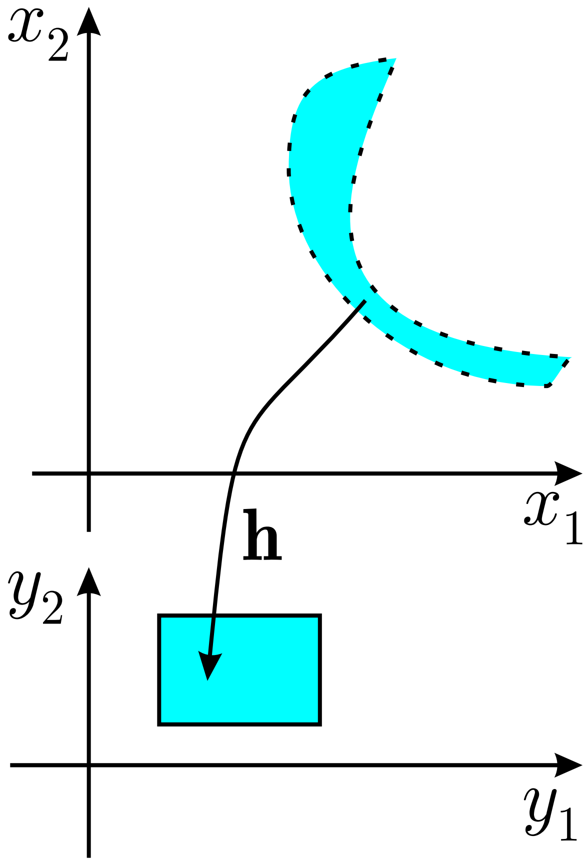


Idealized recursive state estimation



Prediction and correction steps alternate

2.4 Correction step



Set-inversion problem :

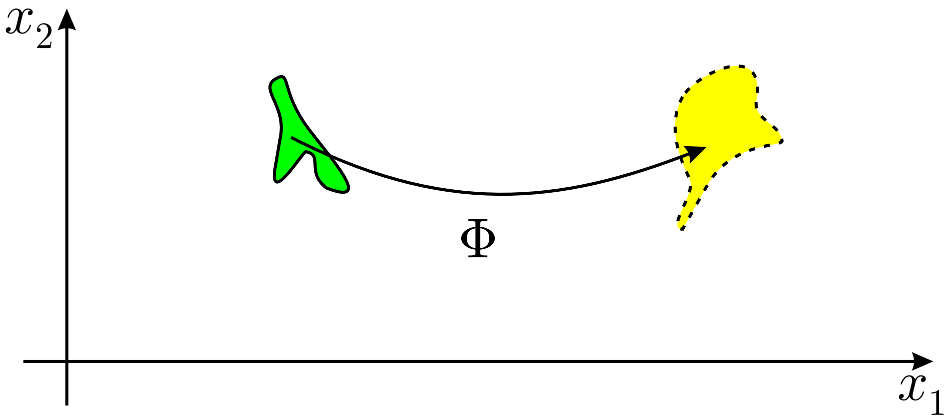
Find

$$\mathcal{X}^O(t_i) = \mathbf{h}^{-1}(\mathbf{y}(t_i) - [\mathbf{v}(t_i)]) .$$

Solution provided by Sivia.

(similar to the discrete-time case)

2.5 Prediction step

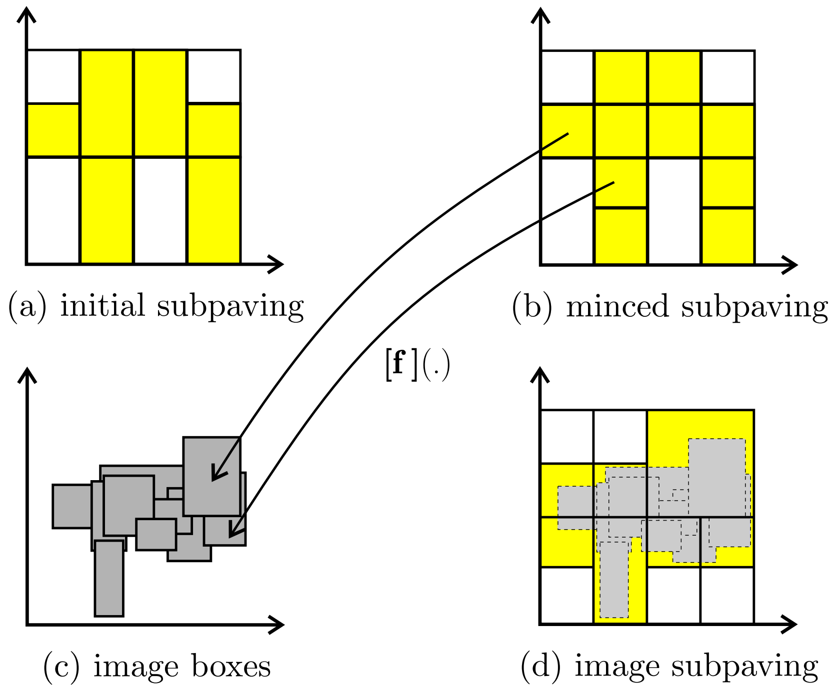


Flow ϕ difficult to obtain in general.

Situation much simpler with discrete-time state equation

$$\mathbf{x}(k+1) = \mathbf{f}_k(\mathbf{x}_k, \mathbf{w}, \mathbf{u}_k).$$

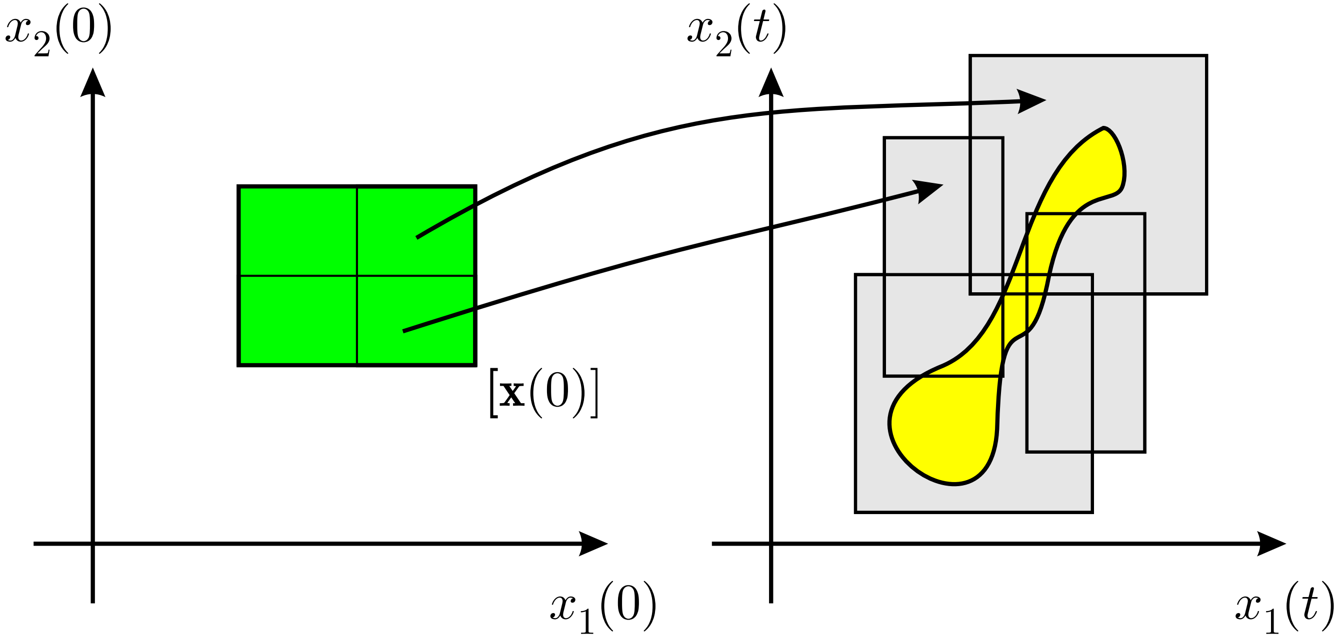
$\Longrightarrow$ ImageSp



Guaranteed numerical integration of continuous-time state equation

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \mathbf{f}\left(\mathbf{x}, \mathbf{w}, \mathbf{u}\right)$$

combined with ImageSp.



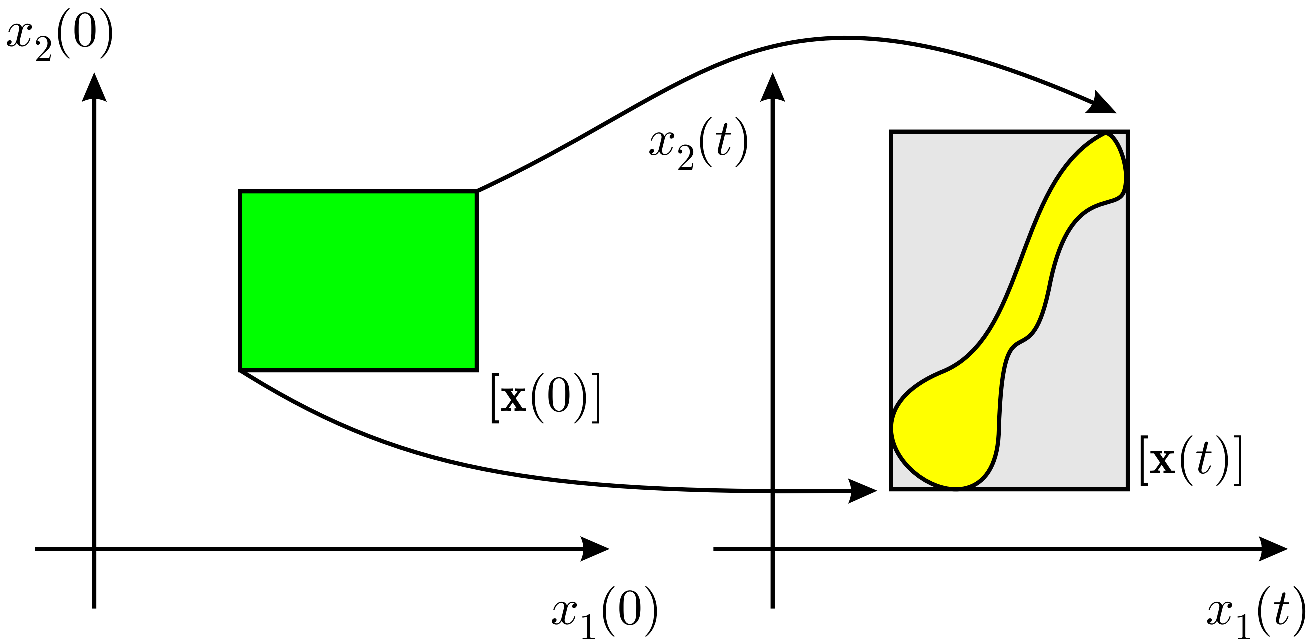
When state equation/initial conditions not well known

$\Longrightarrow$ no accurate box enclosures

Enclosure of state equation between two cooperative systems

+

Guaranteed numerical integration of the cooperative systems



Guaranteed version of Gouzé's interval observer

Accurate box enclosure

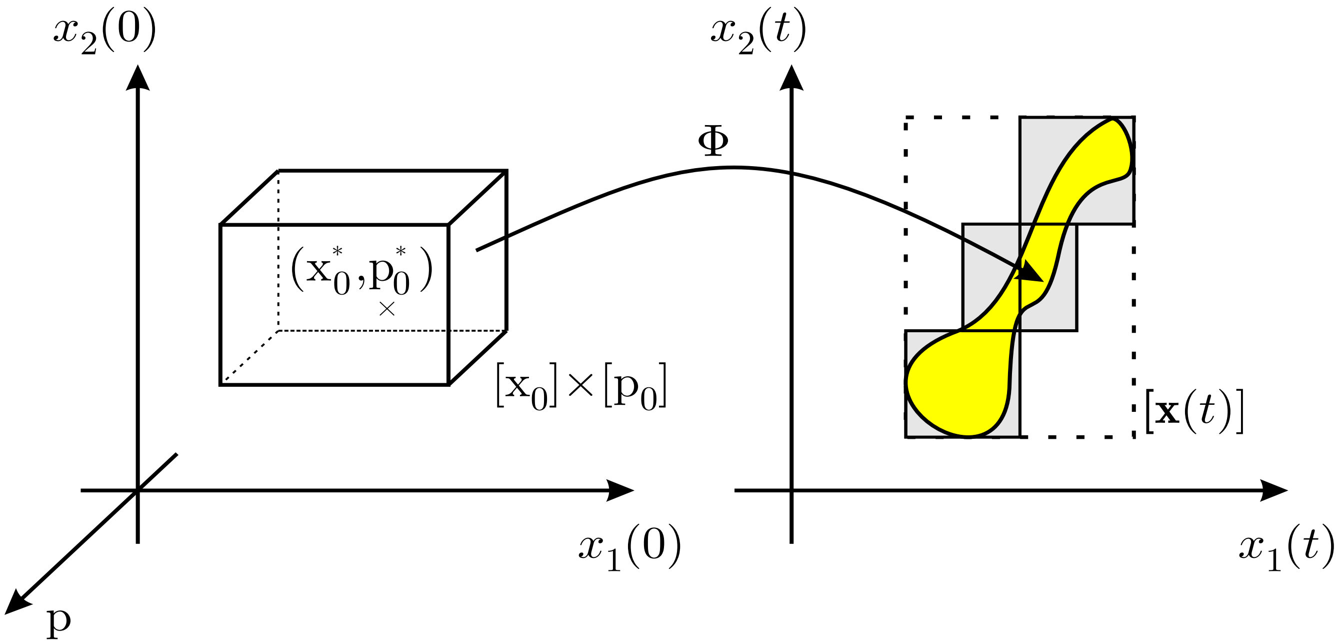
Enclosure of uncertain state equation between punctual dynamical systems

+

Guaranteed numerical integration of the punctual systems

+

IMAGESP



2.6 Bounding the uncertain state equations

Using a reformulation of Müller's theorems (Muller,1926).

Theorem 1 (Existence) *Assume that the function* $\mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{w}, t)$ *is continuous on a domain*

$$\mathbb{T} : \begin{cases} a \leqslant t \leqslant b \\ \boldsymbol{\omega}(t) \leqslant \mathbf{x} \leqslant \boldsymbol{\Omega}(t) \\ \underline{\mathbf{p}}_0 \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}}_0 \\ \underline{\mathbf{w}} \leqslant \mathbf{w}(t) \leqslant \overline{\mathbf{w}} \end{cases}$$

where $\omega_i(t)$ *and* $\Omega_i(t)$, $i = 1 \ldots n_{\mathrm{x}}$, *are continuous on* $[a, b]$ *and such that*

1. $\boldsymbol{\omega}(a) = \underline{\mathbf{x}}_0$ *and* $\boldsymbol{\Omega}(a) = \overline{\mathbf{x}}_0$,
2. *for* $i = 1 \ldots n_{\mathrm{x}}$,

$$D^{\pm}\omega_i(t) \leqslant \min_{\underline{\mathbb{T}}_i(t)} f_i(\mathbf{x}, \mathbf{p}, \mathbf{w}, t) \text{ and } D^{\pm}\Omega_i(t) \geqslant \max_{\overline{\mathbb{T}}_i(t)} f_i(\mathbf{x}, \mathbf{p}, \mathbf{w}, t),$$

where $\underline{\mathbb{T}}_i(t)$ *and* $\overline{\mathbb{T}}_i(t)$ *are subsets of* $\mathbb{T}$ *defined by*

$$\underline{\mathbb{T}}_i(t) : \begin{cases} t = t, \\ x_i = \omega_i(t), \\ \omega_j(t) \leqslant x_j \leqslant \Omega_j(t), \ j \neq i, \\ \underline{\mathbf{p}}_0 \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}}_0, \\ \underline{\mathbf{w}} \leqslant \mathbf{w}(t) \leqslant \overline{\mathbf{w}}, \end{cases} \qquad \overline{\mathbb{T}}_i(t) : \begin{cases} t = t, \\ x_i = \Omega_i(t), \\ \omega_j(t) \leqslant x_j \leqslant \Omega_j(t), \ j \neq i, \\ \underline{\mathbf{p}}_0 \leqslant \mathbf{p} \leqslant \overline{\mathbf{p}}_0, \\ \underline{\mathbf{w}} \leqslant \mathbf{w}(t) \leqslant \overline{\mathbf{w}}. \end{cases}$$

Then, for any $\mathbf{x}(0) \in [\underline{\mathbf{x}}_0, \overline{\mathbf{x}}_0]$, $\mathbf{p} \in \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right]$ *and* $\mathbf{w}(t)$ *a solution to the dynamical system exists, which remains in*

$$\mathbb{E} : \begin{cases} a \leqslant t \leqslant b \\ \boldsymbol{\omega}(t) \leqslant \mathbf{x} \leqslant \boldsymbol{\Omega}(t) \end{cases}$$

and equals $\mathbf{x}(0)$ *at* $t = 0$. $\diamond$

Theorem 2 (Uniqueness) *Moreover, if for any* $\mathbf{p} \in \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right]$ *and* $\mathbf{w}(t)$ *satisfying* $\underline{\mathbf{w}} \leqslant \mathbf{w}(t) \leqslant \overline{\mathbf{w}}$ *at any* $t \in [a, b]$,

$$\mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{w}, t) \text{ is Lipschitz with respect to } \mathbf{x} \text{ over } \mathbb{D},$$

then for any given $\mathbf{x}(0)$, $\mathbf{p}$ *and* $\mathbf{w}(t)$*, this solution is unique.* $\diamondsuit$

Specific version when $\mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{w}, t)$ satisfies condition close to cooperativity.

Theorem 3 (cooperative) *Assume that the function $\mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{w}, t)$ is continuous on a domain $\mathbb{T}'$ that is the same as $\mathbb{T}$ in Theorem 1 where $\omega_i(t)$ and $\Omega_i(t)$ are continuous over $[a, b]$ for $i = 1 \ldots n_{\mathrm{x}}$ and such that*

1. $\boldsymbol{\omega}(a) = \underline{\mathbf{x}}_0$ *and* $\boldsymbol{\Omega}(a) = \overline{\mathbf{x}}_0$,
2. *for* $i = 1 \ldots n_{\mathrm{x}}$,

$$D^{\pm}\omega_i(t) \leqslant \min_{\underline{\mathbb{T}}'_i(t)} f_i(\mathbf{x}, \mathbf{p}, \mathbf{v}, t) \text{ and } D^{\pm}\Omega_i(t) \geqslant \max_{\overline{\mathbb{T}}'_i(t)} f_i(\mathbf{x}, \mathbf{p}, \mathbf{v}, t),$$

where $\underline{\mathbb{T}}'_i(t)$ and $\overline{\mathbb{T}}'_i(t)$ are subsets of $\mathbb{T}$ defined by

$$\begin{aligned} \underline{\mathbb{T}}'_i(t) &= \{\boldsymbol{\omega}(t)\} \times \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right] \times [\underline{\mathbf{w}}, \overline{\mathbf{w}}] \times \{t\} \\ \overline{\mathbb{T}}'_i(t) &= \{\boldsymbol{\Omega}(t)\} \times \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right] \times [\underline{\mathbf{w}}, \overline{\mathbf{w}}] \times \{t\}. \end{aligned}$$

Assume that, if for all $t \in [a, b]$ and $(\mathbf{x}, \mathbf{y}) \in [\boldsymbol{\omega}(t), \boldsymbol{\Omega}(t)]^{\times 2}$,

$$x_i \leqslant y_i, i = 1 \ldots n_x, i \neq j$$
$$\Downarrow$$
$$f_j(\mathbf{x}, \mathbf{p}, \mathbf{w}, t) \leqslant f_j(\mathbf{y}, \mathbf{p}, \mathbf{w}, t), \; j = 1 \ldots n_x$$

for all $\mathbf{p} \in \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right]$, $\mathbf{w} \in [\underline{\mathbf{w}}, \overline{\mathbf{w}}]$.

Then, for any $\mathbf{x}(0) \in [\underline{\mathbf{x}}_0, \overline{\mathbf{x}}_0]$, $\mathbf{p} \in \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right]$ and $\mathbf{w}(t)$, the solution of the dynamical system exists and remains in $\mathbb{E}$ and equals $\mathbf{x}(0)$ at $t = 0$.

The uniqueness conditions are the same as in Theorem 1. $\diamondsuit$

Inclusion function for $\boldsymbol{\phi}(t)$:

$$[\boldsymbol{\phi}](t) = [\boldsymbol{\omega}(t), \boldsymbol{\Omega}(t)].$$

2.7 Obtaining $\boldsymbol{\omega}(t)$ and $\boldsymbol{\Omega}(t)$

- No cooperativity conditions.
  Build a system

$$\begin{cases} \underline{\mathbf{x}}' = \underline{\mathbf{g}}_1(\underline{\mathbf{x}},\overline{\mathbf{x}},\underline{\mathbf{p}}_0,\overline{\mathbf{p}}_0,\underline{\mathbf{w}}(t),\overline{\mathbf{w}}(t),t),\ \underline{\mathbf{x}}(0)=\underline{\mathbf{x}}_0, \\ \overline{\mathbf{x}}' = \overline{\mathbf{g}}_1(\underline{\mathbf{x}},\overline{\mathbf{x}},\underline{\mathbf{p}}_0,\overline{\mathbf{p}}_0,\underline{\mathbf{w}}(t),\overline{\mathbf{w}}(t),t),\ \overline{\mathbf{x}}(0)=\overline{\mathbf{x}}_0, \end{cases}$$

the solution $\left(\boldsymbol{\omega}_1^{\mathrm{T}}(t),\boldsymbol{\Omega}_1^{\mathrm{T}}(t)\right)^{\mathrm{T}}$ of which satisfies the requirements of Theorem 1.

- Cooperativity conditions.
  Build two systems

$$\begin{array}{rcl}\underline{\mathbf{x}}' & = & \underline{\mathbf{g}}_2\left(\underline{\mathbf{x}}, \underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0, \underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t), t\right),\ \underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \\ \overline{\mathbf{x}}' & = & \overline{\mathbf{g}}_2\left(\overline{\mathbf{x}}, \underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0, \underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t), t\right),\ \overline{\mathbf{x}}(0) = \overline{\mathbf{x}}_0\end{array}$$

such that for all $t \in [a, b]$, $\mathbf{x} \in \mathbb{D}$, $\mathbf{p} \in \left[\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0\right]$ and $\mathbf{w} \in [\underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t)]$ one has

$$\underline{\mathbf{g}}_2\left(\mathbf{x}, \underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0, \underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t), t\right) \leqslant \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{w}, t) \tag{1}$$

and

$$\overline{\mathbf{g}}_2\left(\mathbf{x}, \underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0, \underline{\mathbf{w}}(t), \overline{\mathbf{w}}(t), t\right) \geqslant \mathbf{f}(\mathbf{x}, \mathbf{p}, \mathbf{w}, t). \tag{2}$$

Then the solutions $\boldsymbol{\omega}_2(t)$ and $\boldsymbol{\Omega}_2(t)$ of these two EDOs satisfy the conditions required by Theorem 3.

Example 1 *When state equation can be written as*

$$\mathbf{x}' = \mathbf{f}_0(\mathbf{x}, \mathbf{p}, t) + \mathbf{w}(t)$$

and when the components $f_{0,i}(\mathbf{x}, \mathbf{p}, t)$, $i = 1 \dots n_{\mathrm{x}}$ *of* $\mathbf{f}_0(\mathbf{x}, \mathbf{p}, t)$ *are monotonic with respect to* $\mathbf{x}$, $\mathbf{p}$ *and* $\mathbf{v}$, *except to* x_i, *the functions* $\underline{\mathbf{g}}_1$, $\overline{\mathbf{g}}_1$, $\underline{\mathbf{g}}_2$ *and* $\overline{\mathbf{g}}_2$ *are easy to define. For example, to build* $\underline{g}_{1,i}$, *in the formal expression of* $f_{0,i}(\mathbf{x}, \mathbf{p}, t)$, *replace*

1. x_i *by* $\underline{x}_i$,
2. *for* $j \neq i$, x_j *by* $\overline{x}_j$ *if* $\frac{\partial f_{0,i}}{\partial x_j} \leqslant 0$ *and by* $\underline{x}_j$ *if* $\frac{\partial f_{0,i}}{\partial x_j} \geqslant 0$ *for all* $t \in [a, b]$, $\mathbf{x} \in \mathbb{D}$ *and* $\mathbf{p} \in [\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0]$,
3. *for* $k = 1 \dots n_{\mathrm{p}}$, p_k *by* $\overline{p}_k$ *if* $\frac{\partial f_{0,i}}{\partial p_k} \leqslant 0$ *for all* $t \in [a, b]$, $\mathbf{x} \in \mathbb{D}$ *and* $\mathbf{p} \in [\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0]$ *and by* $\underline{p}_k$ *if* $\frac{\partial f_{0,i}}{\partial p_k} \geqslant 0$ *for all* $t \in [a, b]$, $\mathbf{x} \in \mathbb{D}$ *and* $\mathbf{p} \in [\underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0]$.

At last, add $\underline{w}_i(t)$ *to the obtained expression. A similar construction of* $\overline{g}_{1,i}$ *may be performed, but with reversed monotonicity conditions.*

2.8 Examples

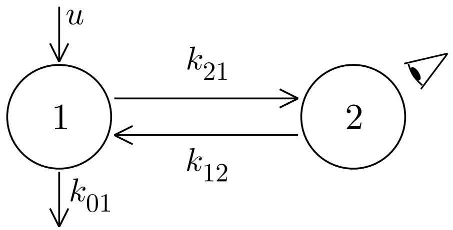


Figure 1: Two-compartment model

Parameters k_{12} and k_{21} constant. Parameter k_{01} depends nonlinearly of the quantity of material present in first compartment

$$k_{01} = \frac{p_1}{p_2 + x_1}.$$

Then

$$\mathbf{p} = (p_1, p_2, k_{12}, k_{21})^{\mathrm{T}}.$$

Quantities of material in both compartments evolve according to

$$\mathbf{x}' = \begin{pmatrix} -p_4 x_1 - \dfrac{p_1 x_1}{1 + p_2 x_1} + p_3 x_2 + u \\ p_4 x_1 - p_3 x_2 \end{pmatrix} \qquad (3)$$

Initial state vector $\mathbf{x}_0 = (1, 0)$, assumed to be known.

For all $t \geqslant 0$, $u(t) = 0$.

No state perturbation is considered.

$\mathbf{p}$ is only assumed to belong to

$$[\mathbf{p}_0] = [0.9, 1.1] \times [1.1, 1.3] \times [0.45, 0.55] \times [0.2, 0.3].$$

Only content of second compartment is measured,

$$y_{\mathrm{m}}(t_k) = x_2(t_k) + v(t_k).$$

The evolution of the state of (3) is studied for $t \in [0, 10]$.

2.8.1 Using only prediction

State equation is cooperative:

Lower dynamical system

$$\underline{\mathbf{x}}' = \underline{\mathbf{g}}_c\left(\underline{\mathbf{x}}, \underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0, t\right), \ \underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \tag{4}$$

with

$$\underline{\mathbf{g}}_c(\cdot) = \begin{pmatrix} -\overline{p}_4\underline{x}_1 - \dfrac{\overline{p}_1\underline{x}_1}{1+\underline{p}_2\underline{x}_1} + \underline{p}_3\underline{x}_2 + u \\ \underline{p}_4\underline{x}_1 - \overline{p}_3\underline{x}_2 \end{pmatrix}$$

Upper dynamical system

$$\overline{\mathbf{x}}' = \overline{\mathbf{g}}_c\left(\overline{\mathbf{x}}, \underline{\mathbf{p}}_0, \overline{\mathbf{p}}_0, t\right), \ \overline{\mathbf{x}}(0) = \overline{\mathbf{x}}_0 \tag{5}$$

with

$$\overline{\mathbf{g}}_c(\cdot) = \begin{pmatrix} -\underline{p}_4\overline{x}_1 - \dfrac{\underline{p}_1\overline{x}_1}{1+\overline{p}_2\overline{x}_1} + \overline{p}_3\overline{x}_2 + u \\ \overline{p}_4\overline{x}_1 - \underline{p}_3\overline{x}_2 \end{pmatrix}$$

With $\underline{\mathbf{x}}_0 = \overline{\mathbf{x}}_0 = \mathbf{x}_0$, Müller's theorem is satisfied.

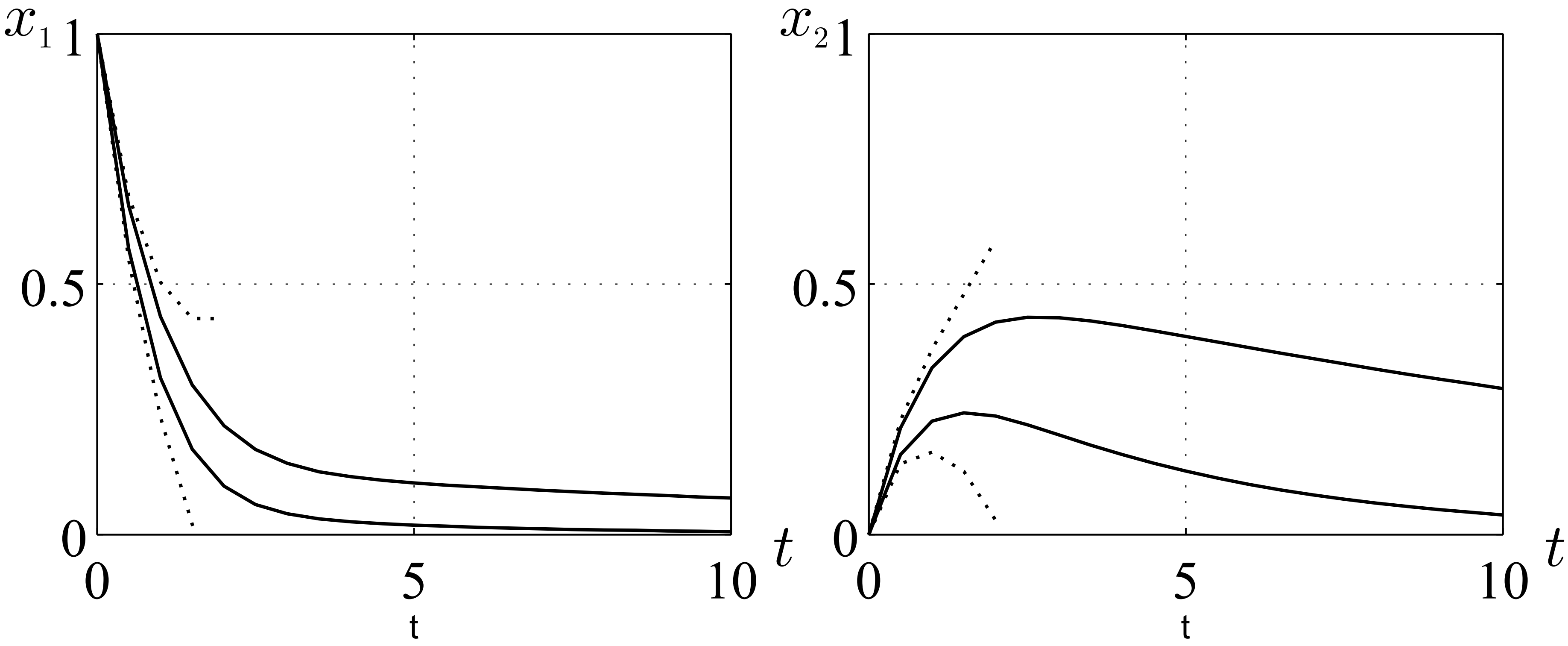


Figure 2: Evolution of the state estimate using prediction only: direct numerical integration (dotted lines) and using cooperativity or its variant of Theorem 3 (bold lines)

2.8.2 Taking measurements into account

Measurements are now available.

Data taken every 2 s, corrupted by an additive noise in $[-0.05, 0.05]$.

$[v_k] = [-0.05, 0.05]$.

Table 1: Noisy data used for state estimation

t_k	2	4	6	8	10
$y(t_k)$	0.323	0.278	0.145	0.186	0.079

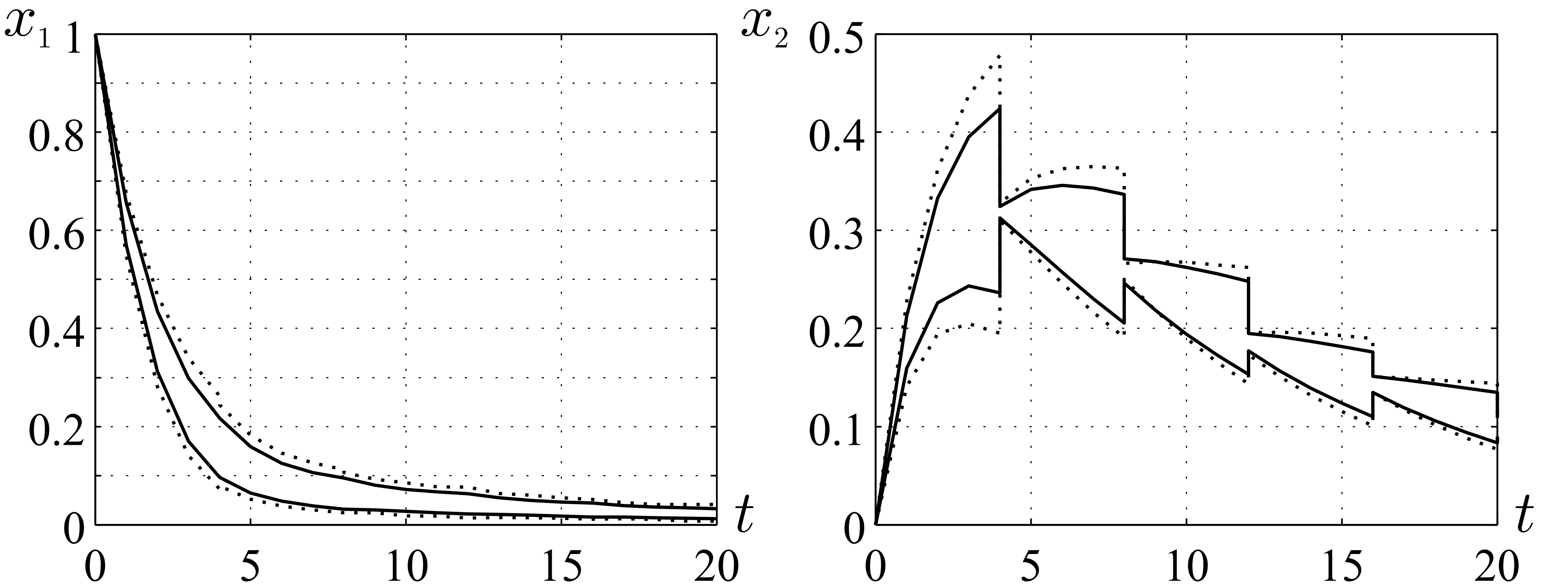


Figure 3: Comparison of two state estimation algorithm: SE-DNI (dotted lines) and SE-MY (bold line)

Computing times:

- 70 s for SE-DNI (direct numerical integration)
- 10 s for SE-MT (Müller's theorem)

2.8.3 Joint parameter and state estimation

Now p_4 added to the state vector $\mathbf{x}$

$$\mathbf{x}_\mathrm{e} = \left(\mathbf{x}^\mathrm{T}, p_4\right)^\mathrm{T}.$$

Assume p_4 constant: extended dynamic

$$\mathbf{x}'_\mathrm{e} = \begin{pmatrix} -x_{\mathrm{e}3}x_{\mathrm{e}1} - \dfrac{p_1 x_{\mathrm{e}1}}{1 + p_2 x_{\mathrm{e}1}} + p_3 x_{\mathrm{e}2} + u \\ x_{\mathrm{e}3}x_{\mathrm{e}1} - p_3 x_{\mathrm{e}2} \\ 0 \end{pmatrix} \tag{6}$$

with $\underline{\mathbf{x}}_{\mathrm{e},0}^\mathrm{T} = \left(\underline{\mathbf{x}}_0^\mathrm{T}, \underline{p}_4\right)$ and $\overline{\mathbf{x}}_{\mathrm{e},0}^\mathrm{T} = \left(\overline{\mathbf{x}}_0^\mathrm{T}, \overline{p}_4\right)$.

New parameter vector

$$\mathbf{q} = \left(p_1, p_2, p_3\right)^\mathrm{T} \in \left(\left[p_1\right], \left[p_2\right], \left[p_3\right]\right)^\mathrm{T}.$$

Extended state equation not cooperative:

Coupled pair of dynamical systems

$$\begin{cases} \underline{\mathbf{x}}_{\mathrm{e}}' = \underline{\mathbf{g}}_{\mathrm{nc}}\left(\underline{\mathbf{x}}_{\mathrm{e}}, \overline{\mathbf{x}}_{\mathrm{e}}, \underline{\mathbf{q}}_0, \overline{\mathbf{q}}_0, t\right), \ \underline{\mathbf{x}}_{\mathrm{e}}(0) = \underline{\mathbf{x}}_{\mathrm{e},0}, \\ \overline{\mathbf{x}}_{\mathrm{e}}' = \overline{\mathbf{g}}_{\mathrm{nc}}\left(\underline{\mathbf{x}}_{\mathrm{e}}, \overline{\mathbf{x}}_{\mathrm{e}}, \underline{\mathbf{q}}_0, \overline{\mathbf{q}}_0, t\right), \ \overline{\mathbf{x}}_{\mathrm{e}}(0) = \overline{\mathbf{x}}_{\mathrm{e},0}, \end{cases} \tag{7}$$

with

$$\underline{\mathbf{g}}_{\mathrm{nc}}(\cdot) = \begin{pmatrix} -\overline{x}_{\mathrm{e}3}\underline{x}_{\mathrm{e}1} - \dfrac{\overline{p}_1\underline{x}_{\mathrm{e}1}}{1+\underline{p}_2\underline{x}_{\mathrm{e}1}} + \underline{p}_3\underline{x}_{\mathrm{e}2} + u \\ \underline{x}_{\mathrm{e}3}\underline{x}_{\mathrm{e}1} - \overline{p}_3\underline{x}_{\mathrm{e}2} \\ 0 \end{pmatrix}$$

and

$$\overline{\mathbf{g}}_{\mathrm{nc}}(\cdot) = \begin{pmatrix} -\underline{x}_{\mathrm{e}3}\overline{x}_{\mathrm{e}1} - \dfrac{\underline{p}_1\overline{x}_{\mathrm{e}1}}{1+\overline{p}_2\overline{x}_{\mathrm{e}1}} + \overline{p}_3\overline{x}_{\mathrm{e}2} + u \\ \overline{x}_{\mathrm{e}3}\overline{x}_{\mathrm{e}1} - \underline{p}_3\overline{x}_{\mathrm{e}2} \\ 0 \end{pmatrix}$$

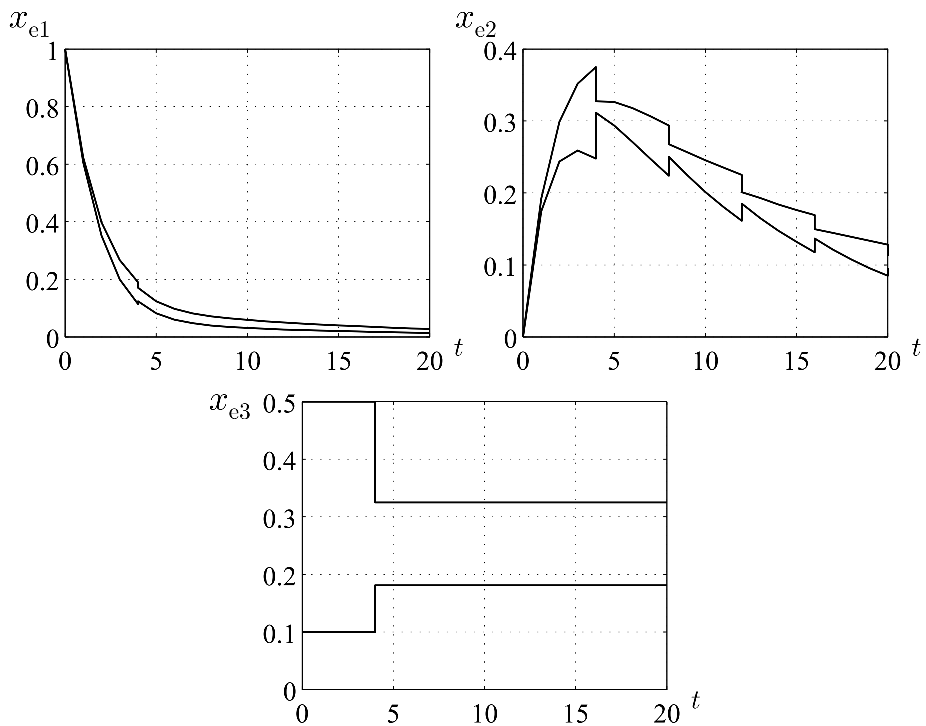
have solutions satisfying Theorem 1.

Data simulated on the same nominal system as before.

Noise corrupting the measurement in $[-0.005, 0.005]$.

All parameters (except p_4) assumed perfectly known.

At $t = 0$, p_4 is only known to belong to $[0.1, 0.5]$.



Takes 40 s on an Athlon at 1.5 GHz.

2.9 Summary of results

1. Recursive state estimation algorithm for continuous-time systems
2. Enclosure of uncertain dynamical system between two point dynamical systems
3. Uncertain parameters or state equations can be considered.
4. Computational complexity compatible with systems of large time constants such as those encountered in biology, pharmacokinetics...

Distributed parameter and state estimation in a network of sensors

Michel Kieffer

L2S - CNRS - SUPELEC - Univ Paris-Sud

March 18, 2009

Wireless sensor networks?

Spatially distributed autonomous devices using sensors connected via a wireless network.

Sensors may be for

- pressure
- temperature
- sound
- vibration
- motion
- ...

Initially developped for military applications (battlefield surveillance)
Now, many civilian applications
(environment monitoring, home automation, traffic control)

[KM04, Hae06]

Applications suggest of many research topics

- protocols for communication between sensors,
- position and localization,
- data compression and aggregation,
- security,
- ...

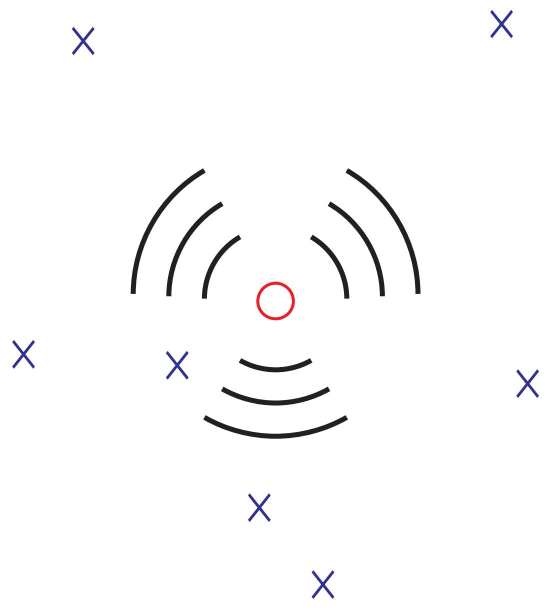
Constraints on WSN

- limited computing capabilities,
- limited communication capacity,
- power consumption restricted.

Example: WSN for source tracking

Applications

- mobile phone localization and tracking
- computer localization in an ad-hoc network
- co-localisation in a team of robots
- speaker localization
- ...



Source (o) and sensors (x)

Usual methods based on

- Time of arrival
  $\hookrightarrow$ requires good clock synchronization
- Time difference of arrival
  $\hookrightarrow$ sensors cannot work independently
- Angle of arrival
  $\hookrightarrow$ difficult to obtain
- Readings of signal strengh (RSS)
  $\hookrightarrow$ cheap, but not accurate

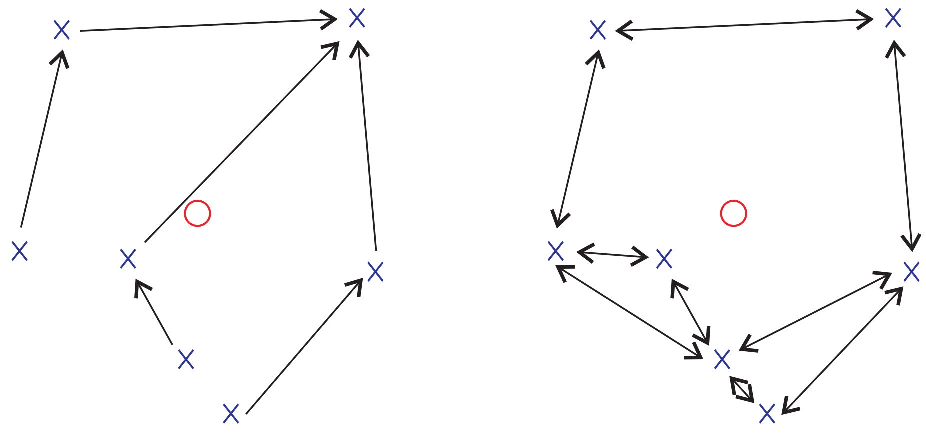
Two localization approaches using RSS

- Centralized

  ↪ all measurements are processed by a unique processing unit

- Distributed

  ↪ measurements are processed by each sensor



Centralized Distributed

Context :

- bounded-error distributed estimation
- interval analysis

Distributed state estimation

Consider a system described by a discrete-time model

$$\mathbf{x}_k = \mathbf{f}_k\left(\mathbf{x}_{k-1}, \mathbf{w}_k, \mathbf{u}_k\right), \tag{1}$$

where

- $\mathbf{x}_k$ state of model at time instant k (sampling period T)
- $\mathbf{w}_k$ state perturbations, assumed bounded in known $[\mathbf{w}]$,
- $\mathbf{u}_k$ input vector, assumed known.

At $k = 0$, $\mathbf{x}_0$ assumed to belong to a set $\mathbb{X}_0$, known.

Assume that each sensor $\ell = 1 \dots L$ of a WSN has access to a measurement

$$\mathbf{y}_k^\ell = \mathbf{g}_k^\ell \left(\mathbf{x}_k, \mathbf{v}_k^\ell\right), \tag{2}$$

where

- $\mathbf{y}_k^\ell$ noisy measurement,
- $\mathbf{v}_k^\ell$ measurement noise, assumed bounded in known $[\mathbf{v}]$,

(1) and (2) are the dynamic and observation equations of the model

Usual measurement equations

$$\begin{aligned} \mathbf{g}_k^\ell \left(\mathbf{x}_k, \mathbf{v}_k^\ell\right) &= \mathbf{h}_k^\ell \left(\mathbf{x}_k\right) + \mathbf{v}_k^\ell \\ \mathbf{g}_k^\ell \left(\mathbf{x}_k, v_k^\ell\right) &= \mathbf{h}_k^\ell \left(\mathbf{x}_k\right) . v_k^\ell \end{aligned}$$

Back to centralized discrete-time state estimation

When all measurements at time k are available at central processing unit, one gets

$$\begin{aligned}\mathbf{x}_k &= \mathbf{f}_k\left(\mathbf{x}_{k-1}, \mathbf{w}_k, \mathbf{u}_k\right),\\ \mathbf{y}_k &= \mathbf{g}_k\left(\mathbf{x}_k, \mathbf{v}_k\right),\end{aligned}$$

with $\mathbf{y}_k^{\mathrm{T}} = \left(\left(\mathbf{y}_k^1\right)^{\mathrm{T}}, \ldots, \left(\mathbf{y}_k^L\right)^{\mathrm{T}}\right)$ and $\mathbf{v}_k^{\mathrm{T}} = \left(\left(\mathbf{v}_k^1\right)^{\mathrm{T}}, \ldots, \left(\mathbf{v}_k^L\right)^{\mathrm{T}}\right)$.

Standard (centralized) state estimation problem.

Information available at time k

$$\mathcal{I}_k = \left\{ \mathbb{X}_0, \{[\mathbf{w}_j]\}_{j=1}^{k}, \{[\mathbf{v}_j]\}_{j=1}^{k}, \{[\mathbf{y}_j]\}_{j=1}^{k} \right\}.$$

Centralized bounded-error state estimate at time k:

set $\mathbb{X}_{k|k}$ of all values of $\mathbf{x}_k$ that are consistent with (1), (2) and $\mathcal{I}_k$.

Idealized algorithm

1. Prediction step

$$\mathbb{X}_{k|k-1} = \left\{ \mathbf{f}_k \left(\mathbf{x}, \mathbf{w}, \mathbf{u}_k \right) \mid \mathbf{x} \in \mathbb{X}_{k-1|k-1}, \ \mathbf{w} \in [\mathbf{w}] \right\}$$

2. Correction step

$$\mathbb{X}_{k|k} = \left\{ \mathbf{x} \in \mathbb{X}_{k|k-1} \mid \mathbf{y}_k = \mathbf{g}_k \left(\mathbf{x}, \mathbf{v} \right), \ \mathbf{v} \in [\mathbf{v}]^{\times L} \right\}.$$

Distributed state estimation

Ideally, any sensor ℓ of the WSN should provide

$$\mathbb{X}^{\ell}_{k|k} = \mathbb{X}_{k|k}.$$

Previous work on this topic

- distributed Kalman filtering [Spe79]
  $\hookrightarrow$ linear models, gaussian noise, instantaneous communications
- application to distributed estimation in power systems [LC05]
- distributed estimation in WSN [RGR06]

Hypotheses

Sensor network is entirely connected (necessary condition to have $\mathbb{X}^{\ell}_{k|k} = \mathbb{X}_{k|k}$)

At time k

- each sensor processes own measurement $\mathbf{y}^{\ell}_{k}$.

Between time k and $k+1$

- each sensor ℓ broadcasts estimates $\mathbb{X}^{\ell,r}_{k|k}$
- each sensor ℓ receives and processes $\mathbb{X}^{s,1}_{k|k}$, $s \in \mathcal{C}(\ell)$ where $\mathcal{C}(\ell)$ set of indices of sensors connected to ℓ
  Process may be repeated (several roundtrips).

Before $k+1$,

- each sensor ℓ builds an estimate $\mathbb{X}^{\ell}_{k|k}$.

Proposed idealized algorithm

For sensor ℓ

At time k:

$$\mathbb{X}_{k|k-1}^{\ell} = \left\{ \mathbf{f}_k \left(\mathbf{x}, \mathbf{w}, \mathbf{u}_k \right) \mid \mathbf{x} \in \mathbb{X}_{k-1|k-1}^{\ell},\ \mathbf{w} \in [\mathbf{w}] \right\}.$$

$$\mathbb{X}_{k|k}^{\ell,0} = \left\{ \mathbf{x} \in \mathbb{X}_{k|k-1}^{\ell} \mid \mathbf{y}_k^{\ell} = \mathbf{g}_k^{\ell} \left(\mathbf{x}, \mathbf{v} \right),\ \mathbf{v} \in [\mathbf{v}] \right\}.$$

Between k and $k+1$,
for $r = 1$ to $R_{\max}$ (number of roundtrips)

$$\mathbb{X}_{k|k}^{\ell,r} = \bigcap_{s \in \mathcal{C}(\ell)} \mathbb{X}_{k|k}^{s,,r-1}$$

Just before $k+1$

$$\mathbb{X}_{k|k}^{\ell} = \mathbb{X}_{k|k}^{\ell,R_{\max}}.$$

One may easily prove that

$$\mathbb{X}_{k|k} \subset \mathbb{X}^{\ell}_{k|k}.$$

Non-trivial conditions to have

$$\mathbb{X}_{k|k} = \mathbb{X}^{\ell}_{k|k}$$

are more difficult to obtain. Involve

- network connectivity
- largest distance (in links) between sensors
- ...

Problem studied in [Yok01, BFV$^+$05].

Practical algorithm

Implementation issues:

- Boxes or subpavings used to represent sets,
- Basic interval evaluation or ImageSp [KJW02] for prediction step,
- Interval constraint propagation or Sivia [JW93] for correction step

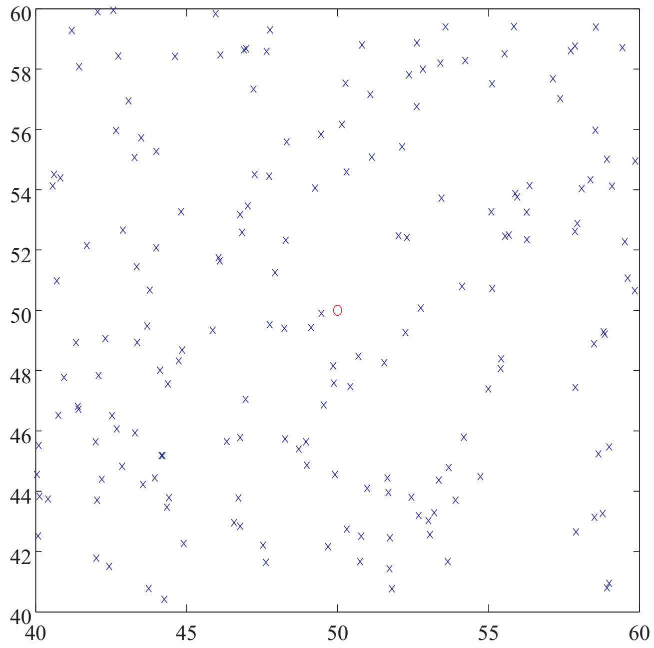
Application: Static source localization

Known sensor locations

$$\mathbf{r}_\ell \in \mathbb{R}^2, \ell = 1 \ldots L$$

Unknown source location

$$\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$$



Source (o) and sensors (x)

Mean power $\overline{P}(d_\ell)$ (in dBm) received by ℓ-th sensor described by Okumura-Hata model

$$\overline{P}_{\mathrm{dB}}(d_\ell) = P_0 - 10 n_{\mathrm{p}} \log \frac{d_\ell}{d_0}, \qquad (3)$$

where

- n_{p} is the path-loss exponent (unknown, but constant)
- $d_\ell = |\mathbf{r}_\ell - \boldsymbol{\theta}|$.

Received power assumed to remain within some known bounds (here)

$$P_{\mathrm{dB}}(d) \in \left[P_0 - 10 n_{\mathrm{p}} \log \frac{d}{d_0} - e, P_0 - 10 n_{\mathrm{p}} \log \frac{d}{d_0} + e \right], \qquad (4)$$

where e is assumed known

$\hookrightarrow$ bounded-error approach.

Bounded-error parameter estimation

RSS by sensor $\ell = 1 \ldots L$

$$y_\ell = h_\ell\left(\boldsymbol{\theta}, A, n_{\mathrm{p}}\right) v_\ell$$

with

$$h_\ell\left(\boldsymbol{\theta}, A, n_{\mathrm{p}}\right) = \frac{A}{\left|\mathbf{r}_\ell - \boldsymbol{\theta}\right|^{n_{\mathrm{p}}}},\ A = 10^{P_0/10} d_0^{n_{\mathrm{p}}}, \tag{5}$$

and

$$v_\ell \in [v] = \left[10^{-e/10}, 10^{e/10}\right].$$

Constant state vector to be estimated

$$\mathbf{x} = \left(A, n_{\mathrm{p}}, \theta_1, \theta_2\right)$$

Distributed approach: interval constraint propagation

At sensor ℓ,

- $y_\ell \in [y_\ell]$, measured
- $\boldsymbol{\theta} \in [\boldsymbol{\theta}]$, obtained from neighbors
- $A \in [A]$, obtained from neighbors
- $n_\mathrm{p} \in [n_\mathrm{p}]$, obtained from neighbors.

Variables must satisfy constraint provided by RSS model

$$y_\ell - \frac{A}{|\mathbf{r}_\ell - \boldsymbol{\theta}|^{n_\mathrm{p}}} = 0. \tag{6}$$

Interval constraint propagation :
reduce the domains for the variables using the constraints

Contracted domains may be written as

$$
\begin{aligned}
[y_\ell'] &= [y_\ell] \cap \frac{[A]}{|\mathbf{r}_\ell - [\boldsymbol{\theta}]|^{[n_\mathrm{p}]}},\\
[A'] &= [A] \cap [y_\ell'] \, |\mathbf{r}_\ell - [\boldsymbol{\theta}]|^{[n_\mathrm{p}]},\\
\left[n_\mathrm{p}'\right] &= [n_\mathrm{p}] \cap (\log([A']) - \log([y_\ell'])) \, / \, \log(|\mathbf{r}_\ell - [\boldsymbol{\theta}]|),\\
[\theta_1'] &= [\theta_1] \cap \left(r_{\ell,1} \pm \sqrt{([A'] \, / \, [y_\ell'])^{2/\left[n_\mathrm{p}'\right]} - (r_{\ell,2} - [\theta_2])^2} \right),\\
[\theta_2'] &= [\theta_2] \cap \left(r_{\ell,2} \pm \sqrt{([A'] \, / \, [y_\ell'])^{2/\left[n_\mathrm{p}'\right]} - (r_{\ell,1} - [\theta_1])^2} \right).
\end{aligned}
$$

Contracted domains still contains all solutions

Results

Networks of $L = 2000$ sensors randomly distributed

Field of 100 m$\times$100 m.

Source

- placed at $\boldsymbol{\theta}^* = (50 \text{ m}, 50 \text{ m})$
- $P_0 = 20$ dBm
- $d_0 = 1$ m
- $n_\text{p} = 2$ (constant over the field)

Measurement noise such that $e = 4$ dBm.

Example of measurements

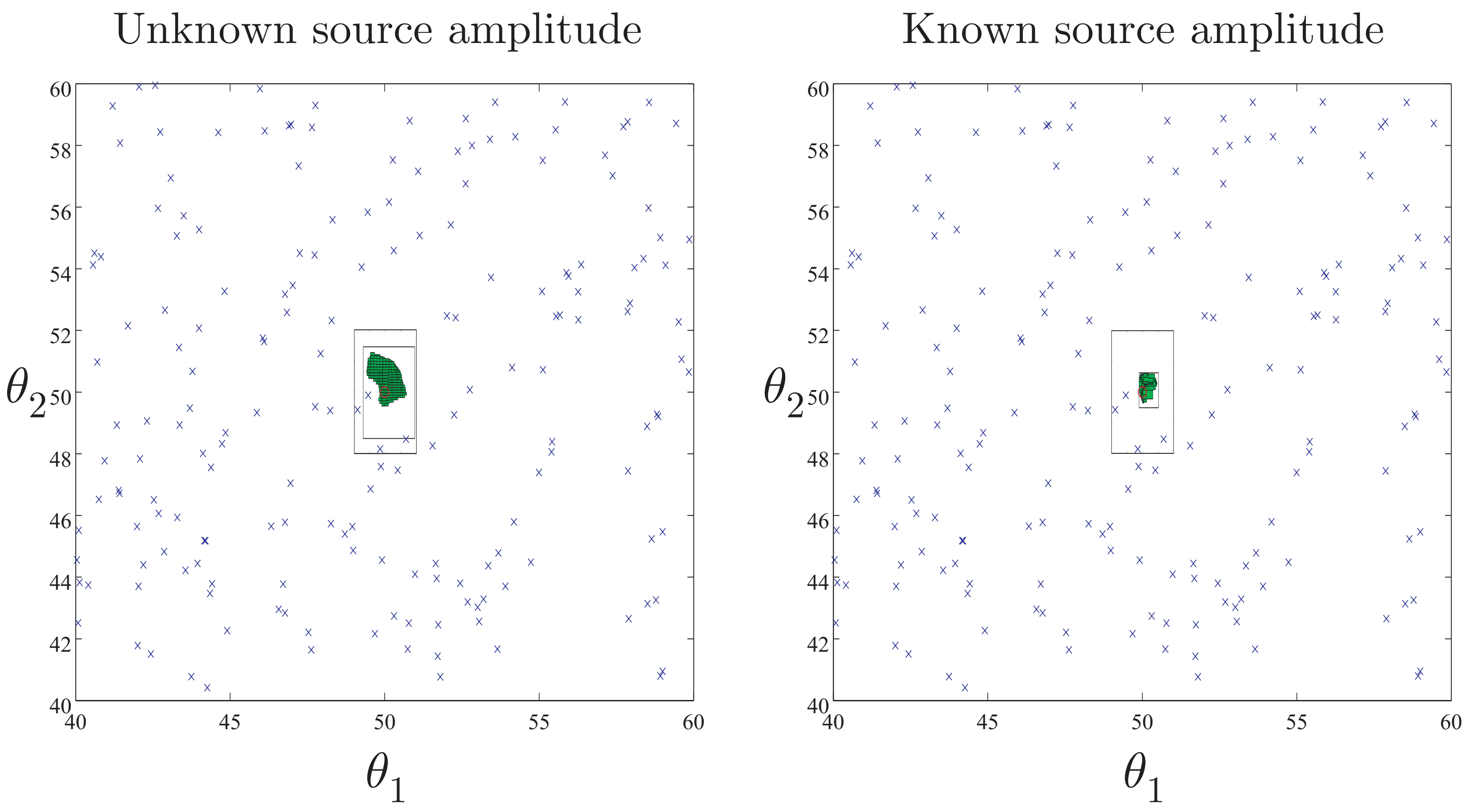
Sensor	68	741	954	⋯
Measurement	$[9.303, 58.698]$	$[17.856, 112.664]$	$[18.644, 117.640]$	⋯

Initial search box (Unknown source amplitude)

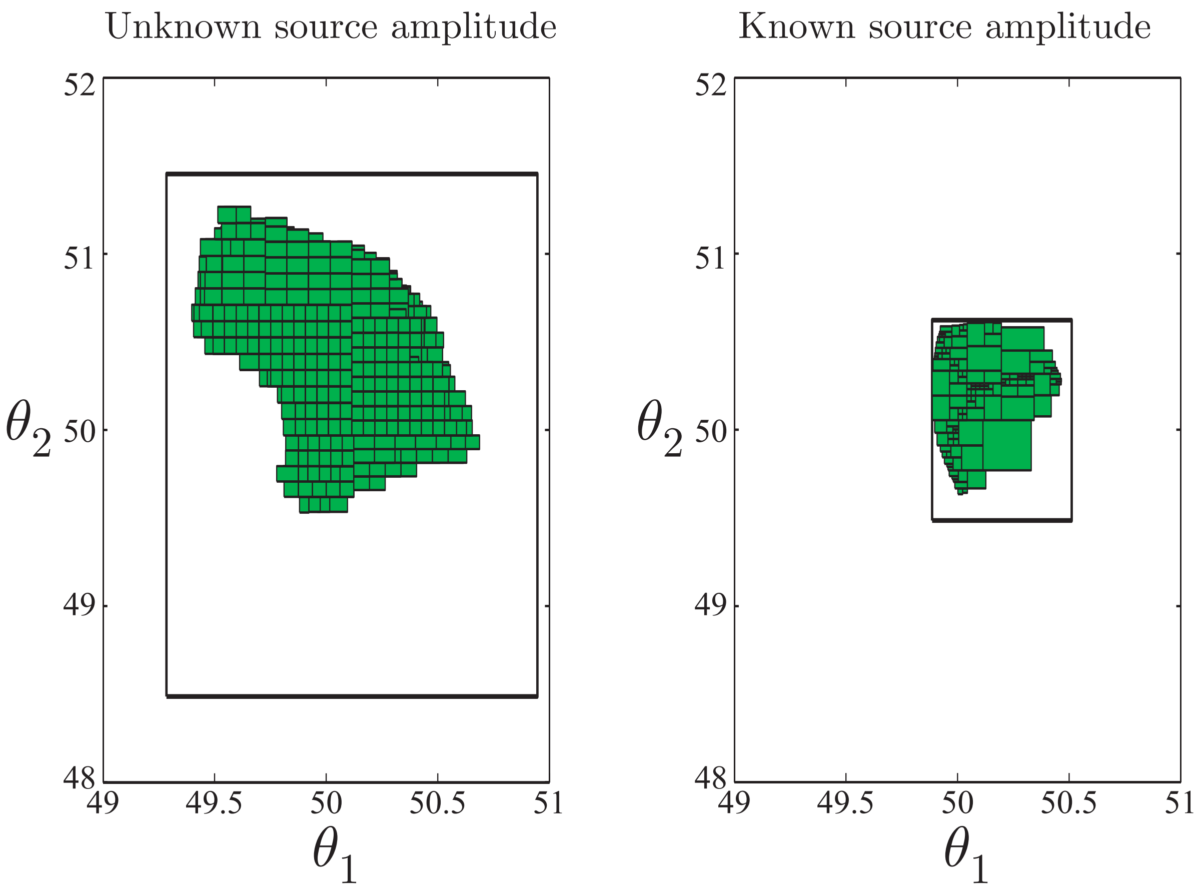
$$[\boldsymbol{\theta}] \times [A] \times [n_{\mathrm{p}}] = [0, 100] \times [0, 100] \times [50, 250] \times [2, 4]$$

Initial search box (Known source amplitude)

$$[\boldsymbol{\theta}] \times [A] \times [n_{\mathrm{p}}] = [0, 100] \times [0, 100] \times [100, 100] \times [2, 4]$$

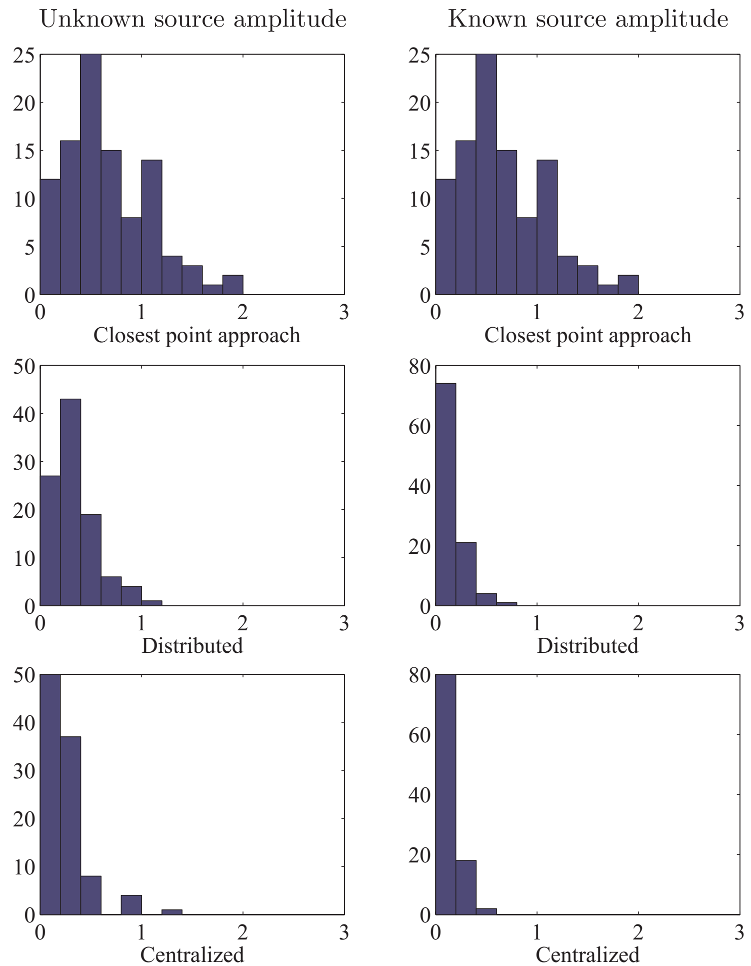


Projection of the solution on the (θ_1, θ_2)-plane



Zoom on the solution

Results – continued



Histograms of estimation error for $\boldsymbol{\theta}$ (100 realizations of sensor field)

Application: Source tracking

Assume now that the source is moving. A and n_{p} assumed to be known.

New state vector

$$\mathbf{x}_k = (\theta_{1,k}, \theta_{2,k}, \phi_{1,k}, \phi_{2,k}, \theta_{1,k-1}, \theta_{2,k-1}, \phi_{1,k-1}, \phi_{2,k-1})^{\mathrm{T}}$$

This long state vector allows to estimate $\left(\phi_{1,k}, \phi_{2,k}\right)$.

State vector evolves according to

$$\begin{pmatrix} \theta_{1,k} \\ \theta_{2,k} \\ \phi_{1,k} \\ \phi_{2,k} \\ \theta_{1,k-1} \\ \theta_{2,k-1} \\ \phi_{1,k-1} \\ \phi_{2,k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_4 & \mathbf{0}_4 \\ \mathbf{I}_4 & \mathbf{0}_4 \end{pmatrix} \begin{pmatrix} \theta_{1,k-1} \\ \theta_{2,k-1} \\ \phi_{1,k-1} \\ \phi_{2,k-1} \\ \theta_{1,k-2} \\ \theta_{2,k-2} \\ \phi_{1,k-2} \\ \phi_{2,k-2} \end{pmatrix} + T. \begin{pmatrix} \phi_{1,k-1} \\ \phi_{2,k-1} \\ w_1 \\ w_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

with $w_1 \in [w]$ and $w_2 \in [w]$.

Contracted domains

$$\left[y'_{\ell,k}\right] = [y_{\ell,k}] \cap \frac{A}{\left|\mathbf{r}_\ell - [\boldsymbol{\theta}_k]\right|^{[n_\mathrm{p}]}},$$

$$\left[\theta'_{1,k}\right] = [\theta_{1,k}] \cap \left(r_{\ell,1} \pm \sqrt{\left(A/\left[y'_{\ell,k}\right]\right)^{2/n_\mathrm{p}} - (r_{\ell,2} - [\theta_{2,k}])^2}\right),$$

$$\left[\theta'_{2,k}\right] = [\theta_{2,k}] \cap \left(r_{\ell,2} \pm \sqrt{\left(A/\left[y'_{\ell,k}\right]\right)^{2/n_\mathrm{p}} - (r_{\ell,1} - [\theta_{1,k}])^2}\right).$$

$$\left[\phi'_{1,k}\right] = \left[\phi_{1,k}\right] \cap \left(\frac{\left[\theta'_{1,k}\right] - \left[\theta'_{1,k}\right]}{T} + T\,[w]\right)$$

$$\left[\phi'_{2,k}\right] = \left[\phi_{2,k}\right] \cap \left(\frac{\left[\theta'_{2,k}\right] - \left[\theta'_{2,k}\right]}{T} + T\,[w]\right)$$

Results

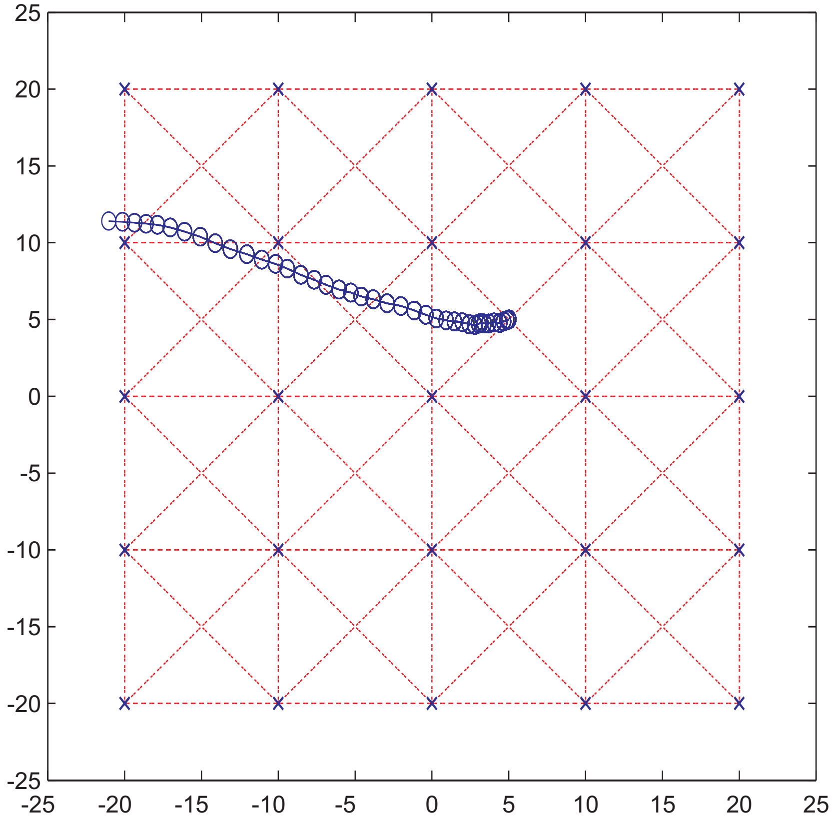
Field of 50 m×50 m (origin at center)

Networks of $L = 25$ sensors (communication range of 15 m)

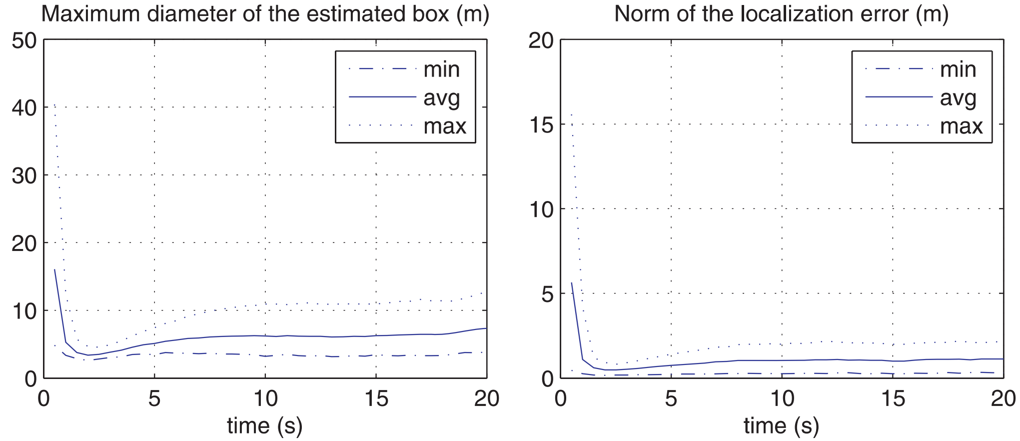
Source

- placed at $\boldsymbol{\theta}^* = (5 \text{ m}, 5 \text{ m})$
- $P_0 = 20$ dBm
- $d_0 = 1$ m
- $n_\text{p} = 2$ (constant over the field)
- $[w] = [-0.5, 0.5]$ m.s^{-2}
- $T = 0.5$ s

Measurement noise such that $e = 4$ dBm.



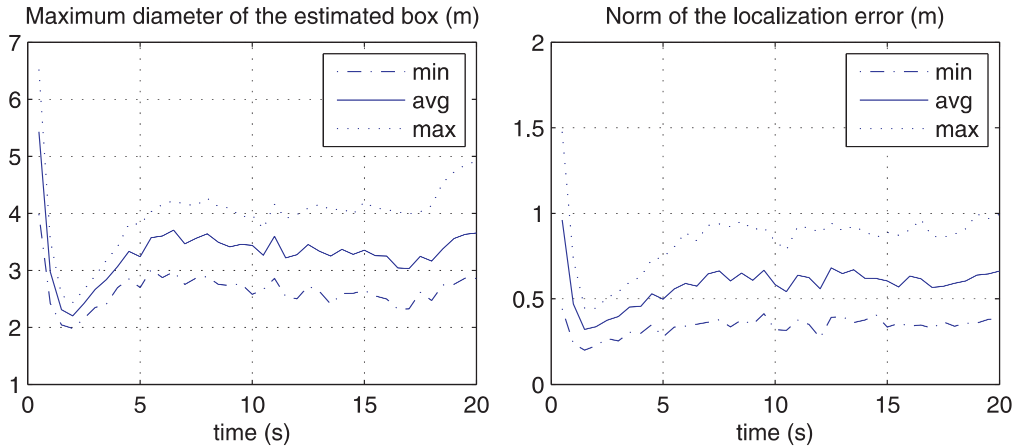
Evolution of the source in the WSN



Localization error and width of the box $[\theta_{1,k}] \times [\theta_{2,k}]$

First iteration

Convergence quite fast



Localization error and width of the box $[\theta_{1,k}] \times [\theta_{2,k}]$

Third iteration

Convergence quite fast

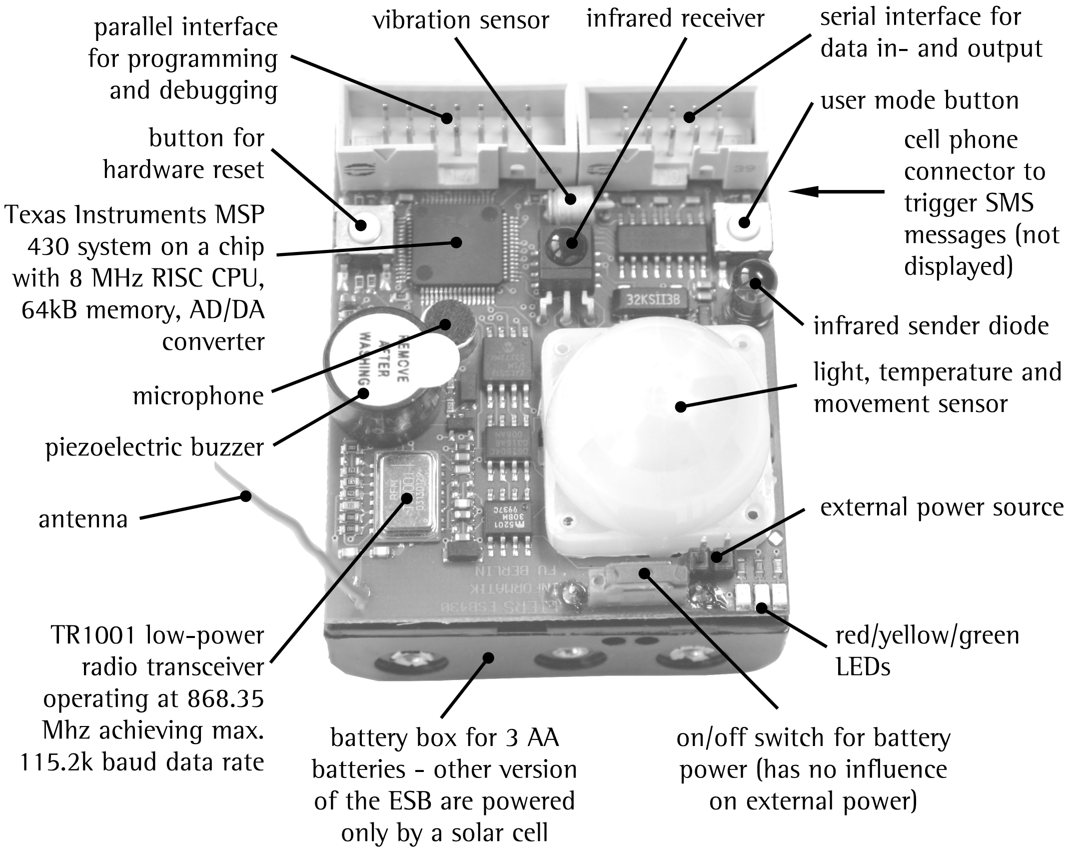
Conclusions

- Distributed source localization
- Estimation in a bounded-error context
  $\hookrightarrow$ guaranteed results (provided hypotheses satisfied)

Further work

- Large space for improvements (compute subpavings and send boxes)
- Robustness to outliers in a distributed context
- Sensor colocalisation (team of robots)
- ...

Work partly supported by NoE NEWCOM++



Overview of the components of the Embedded Sensor Board (ESB) from the FU-Berlin (picture from [Hae06]).

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