Optimal separator for an hyperbola
Application to localization

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Abstract. This paper proposes a minimal contractor and a minimal separator for an area delimited by an hyperbola of the plane. The task is facilitated using actions induced by the hyperoctahedral group of symmetries. An application related to the localization of an object using a TDoA (Time Differential of Arrival) technique is proposed.

1 Introduction

Contractor programming [2] is an efficient tool to solve rigorously complex nonlinear problems involving bounded uncertainties [1] [8] [11]. It is based on the notion of *contractor* which is an operator which shrinks an axis-aligned box $[\mathbf{x}]$ of $\mathbb{R}^n$ without removing any point of the subset $\mathbb{X}$ of $\mathbb{R}^n$ to which it is associated. The set $\mathbb{X}$ is assumed to be defined by equations or inequalities involving the components $x_1, \dots, x_n$ of $\mathbf{x}$. As a result, combined with a paver which bisects boxes, the contractor will allow us to build an outer approximation of the set $\mathbb{X}$. This outer approximation is represented as a list of boxes $\mathcal{L}$ with sides parallel to the coordinate axes. This list $\mathcal{L}$ contains as few elements not from $\mathbb{X}$ as possible. A *separator* [7] is a pair of two contractors: an inner contractor and an outer contractor. A separator for $\mathbb{X}$ with the paver will generate an outer and also an inner approximations of $\mathbb{X}$.

Contractors and separators use abstract domains that are boxes. These operators are well suited to boxes since the intersection (finite or infinite) of boxes is always a box. Contractor programming relies on a catalog of efficient elementary contractors which are usually built using interval arithmetic [10]. There contractors shrink intervals of feasible values for the variables, without removing a single feasible value. Combining all available elementary contractors, we can construct a more sophisticated one consistent with the solution set of the problem we want to solve. This operation introduces a pessimism which has to be balanced by additional bisections performed by the paver. For more efficiency, it is important to extend the catalog by adding some new specific contractors/separators. This is the main objective of this paper where the constraint of interest is hyperbolic (see [6]).

2 Hyperbolic constraint

We consider the quadratic function

$$f(\mathbf{q},\mathbf{x}) = q_0 + q_1x_1 + q_2x_2 + q_3x_1^2 + q_4x_1x_2 + q_5x_2^2 \tag{1}$$

where $\mathbf{q} = (q_0,\ldots,q_5)$ is the known parameter vector and $\mathbf{x} = (x_1,x_2)$ is the vector of variables. The zeros of $f(\mathbf{q},\mathbf{x})$ is a conic section (a circle or other ellipse, a parabola, or an hyperbola). In this paper, we consider the case where the conic section is an hyperbola. We want propose an interval-based method [9] to generate an optimal separator for the set

$$\mathbb{X} = \{(x_1,x_2)|f(\mathbf{q},\mathbf{x}) \leq 0\} \tag{2}$$

in the case where $\mathbf{q}$ is known. The technique is similar to that proposed in [5] for ellipses. This separator will be used to generate an inner and an outer approximations for $\mathbb{X}$. The methodology (see [4]) is based on symmetries of the equation of $f(\mathbf{q},\mathbf{x}) = 0$.

3 Application

We consider an example taken from [3] related to localization. Consider a robot which emits a sound at an unknown time t_0. This sound is received with a delay by three microphones located points $\mathbf{a} : (13,7), \mathbf{b} : (4,6), \mathbf{c} : (16,10)$ of the plane (see Figure 1). Taking into account the time of flight of the sound we want to estimate the position of the object.

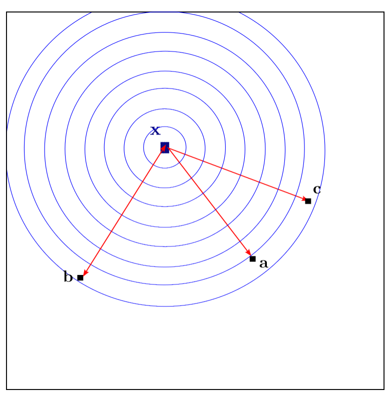

Fig. 1: The robot at position $\mathbf{x}$ emits a sound received later by three microphones $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

We have

$$\begin{array}{rcl} \|\mathbf{x}-\mathbf{a}\| & = & c\cdot(t_a-t_0) \\ \|\mathbf{x}-\mathbf{b}\| & = & c\cdot(t_b-t_0) \\ \|\mathbf{x}-\mathbf{c}\| & = & c\cdot(t_c-t_0) \end{array} \tag{3}$$

where c is the sound speed and t_a, t_b, t_c is the detection time for microphones $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We eliminate t_0 which is unknown to get

$$\begin{array}{rcl} \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{b}\| & = & c\cdot(t_a-t_b)=\ell_{ab} \\ \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{c}\| & = & c\cdot(t_a-t_c)=\ell_{ac} \end{array} \tag{4}$$

The quantities ℓ_{ab}, ℓ_{bc} are called *pseudo-distances*. We assume that we were able to measure the two pseudo distances to get $\ell_{ab} \in [7.9, 8.1]$ and $\ell_{ac} \in [3.9, 4.1]$. The set $\mathbb{X}$ of all feasible locations is defined by

$$\begin{array}{lrcl} \text{(i)} & \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{b}\| & \in & [7.9, 8.1] \\ \text{(ii)} & \|\mathbf{x}-\mathbf{a}\|-\|\mathbf{x}-\mathbf{c}\| & \in & [3.9, 4.1] \end{array} \tag{5}$$

Using a paver, we get an inner and an outer approximations for the solution set. Figure 2 has been generated with a classical forward-backward contractor. We observe a strong clustering effect with many uncertain boxes that the separator is not able to classify.

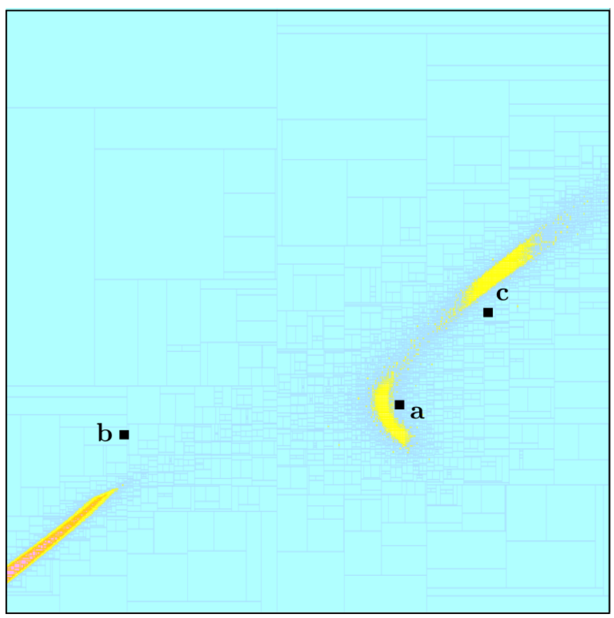

Fig. 2: Set $\mathbb{X}$ of positions using the three microphones (classic)

Figure 3 has been generated using the minimal contractor hyperbola separator. For all figures, the frame box is $[0, 20] \times [0, 20]$ and the accuracy is the same ($\varepsilon = 0.05$). All results are guaranteed since outward rounding is implemented [12]. The clustering effect almost disappeared.

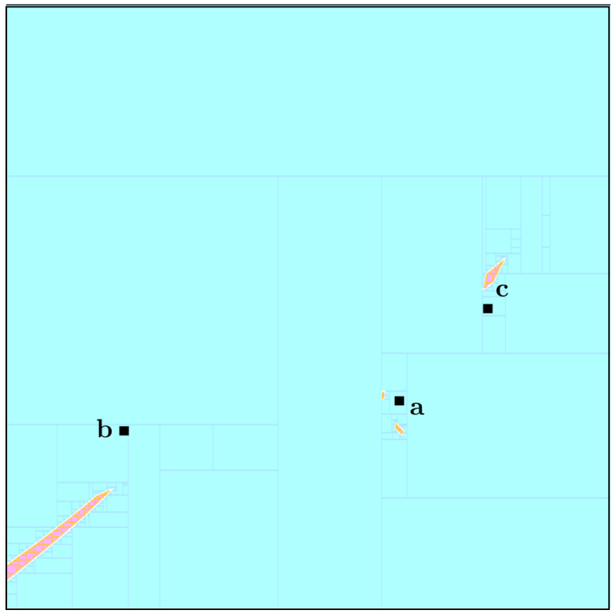

Fig. 3: Set $\mathbb{X}$ of positions using the three microphones (with the hyperbola separator)

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