Set-membership Underwater Localization: Implementing a Separator for the Remoteness Constraint.

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Introduction

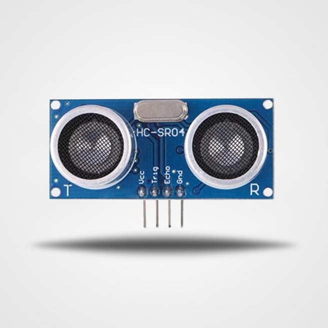


Figure 1: Ultrasonic sensors

State estimation

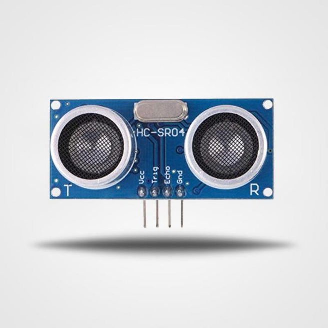
- Essential in robotics
- Based on sensors measurements

Time of Flight sensors

- Directivity cone
- Returns the minimal distance within the measurement's cone

Problem statement

- Deal with uncertainties
- Express this problem as a constraint



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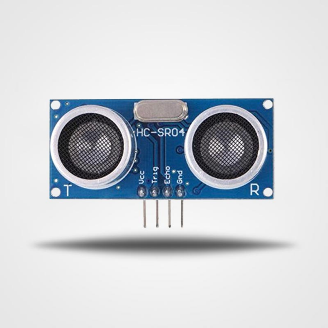

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Contractors & Separators

Definition

A **contractor** $\mathcal{C}_{\mathcal{L}}$ consistent with a constraint $\mathcal{L}$ is an operator that, given a box $[\mathbf{x}] \subseteq \mathbb{IR}^n$, returns a box $\mathcal{C}_{\mathcal{L}}([\mathbf{x}])$ such that:

$$\mathcal{C}_{\mathcal{L}}([\mathbf{x}]) \subseteq [\mathbf{x}] \qquad \text{(Contractance)} \tag{1}$$

$$\mathbf{x} \text{ satisfies } \mathcal{L} \implies \mathbf{x} \in \mathcal{C}_{\mathcal{L}}([\mathbf{x}]) \qquad \text{(Consistency)} \tag{2}$$

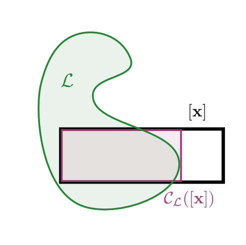

Figure 2: Contractor $\mathcal{C}_{\mathcal{L}}$ for the constraint $\mathcal{L}$.

Definition

A **separator** $\mathcal{S}_{\mathcal{L}}$ consistent with a constraint $\mathcal{L}$ is a pair of contractors $\{\mathcal{S}_{\mathcal{L}}^{in}, \mathcal{S}_{\mathcal{L}}^{out}\}$ on the inside and on the outside of the set, such that:

$$\mathcal{S}_{\mathcal{L}}^{in}([\mathbf{x}]) \cup \mathcal{S}_{\mathcal{L}}^{out}([\mathbf{x}]) = [\mathbf{x}] \quad \text{(Complementarity)} \tag{3}$$

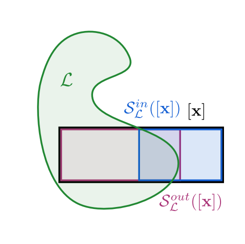

Figure 3: Separator $\mathcal{S}_{\mathcal{L}}$ for the constraint $\mathcal{L}$.

χ-constraint

Definition

The χ-constraint is defined as:

$$\chi(\mathbb{A},\mathbb{B},\mathbb{C}) = \begin{cases} \mathbb{B} & \text{if in } \mathbb{A} \\ \mathbb{C} & \text{otherwise} \end{cases} \tag{4}$$

Implementation

$$\chi(\mathbb{A},\mathbb{B},\mathbb{C}) = (\mathbb{A} \cap \mathbb{B}) \cup (\bar{\mathbb{A}} \cap \mathbb{C}) \tag{5}$$

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Example of sets

$$\begin{aligned} \mathbb{A} &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \\ \mathbb{B} &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \times x_2 \in [-0.15, 0.15]\} \\ \mathbb{C} &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \sqrt{x_1} - x_2 \in [-1.5, 1.5]\} \end{aligned}$$

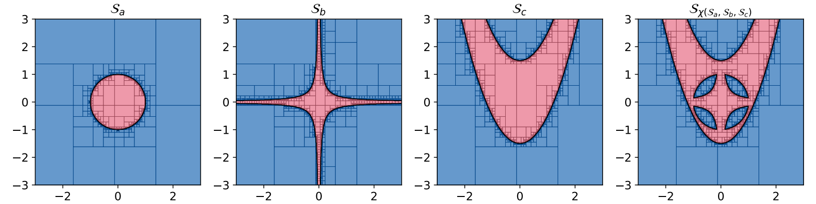


Figure 4: Sets $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, and $\chi(\mathbb{A}, \mathbb{B}, \mathbb{C})$

Remoteness Constraint

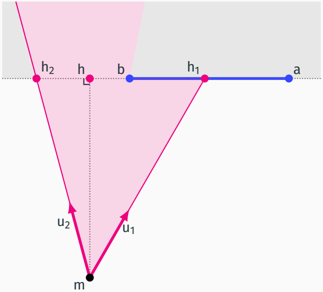


Figure 5: Remoteness of a segment $[\mathbf{a}, \mathbf{b}]$ relative to a point $\mathbf{m}$ and two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Here $d = ||mb||$.

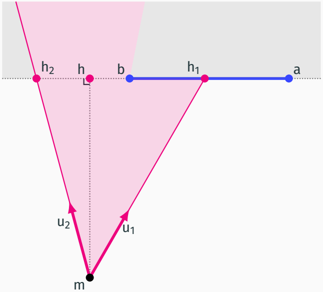
Remoteness Cone

The remoteness cone $\mathcal{C}(\mathbf{m}, \mathbf{u}_1, \mathbf{u}_2)$ is defined by $\mathbf{m}$, $\mathbf{u}_1$ and $\mathbf{u}_2$.

Remoteness Distance

By definition, the remoteness distance d meets

- $d \in \{||\mathbf{ma}||, ||\mathbf{mb}||, ||\mathbf{mh}||, ||\mathbf{mh}_1||, ||\mathbf{mh}_2||, +\infty\}$
- d intersects $\mathcal{C}_{m,a,b}$.



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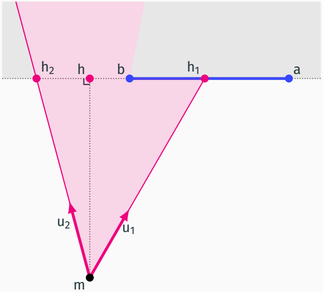

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Remoteness Constraint Examples

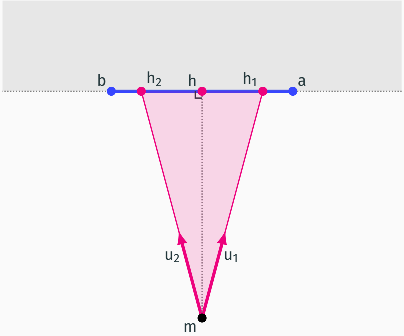


Figure 6: Remoteness $d = ||mh||$.

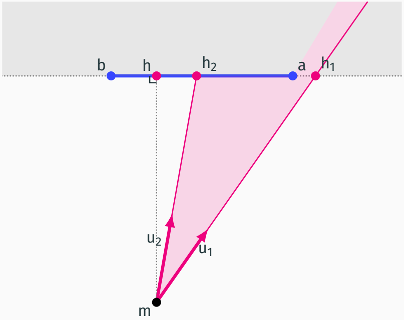


Figure 7: Remoteness $d = ||mh_2||$.

Distance equation

$$||\mathbf{ma}|| = \sqrt{(m_x - a_x)^2 + (m_y - a_y)^2} \quad (6)$$

$$||\mathbf{mb}|| = \sqrt{(m_x - b_x)^2 + (m_y - b_y)^2} \quad (7)$$

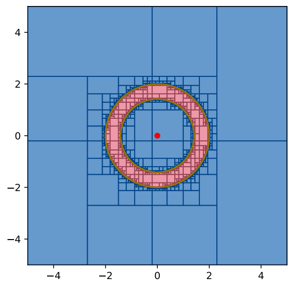


Figure 8: Distance constraint from a point, with $d = [1, 2]$

Orthogonal distance equation

$$||\mathbf{mh}|| = \frac{det(\mathbf{ab}, \mathbf{am})}{||\mathbf{ab}||} \tag{8}$$

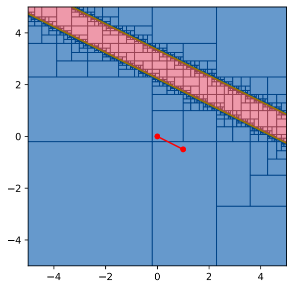


Figure 9: Orthogonal distance constraint with $d = [2, 3]$

Directional distance equation

$$||\mathbf{mh}_1|| = \frac{det(\mathbf{ab},\mathbf{am})}{det(\mathbf{u}_1,\mathbf{ab})} \quad (9)$$

$$||\mathbf{mh}_2|| = \frac{det(\mathbf{ab},\mathbf{am})}{det(\mathbf{u}_2,\mathbf{ab})} \quad (10)$$

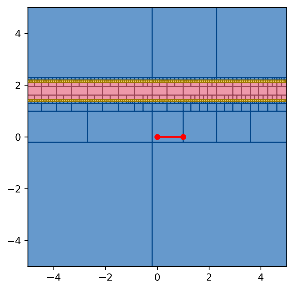


Figure 10: Directional distance constraint with $d = [2, 3]$, and $\mathbf{u} = \begin{bmatrix} cos\left(\frac{\pi}{4}\right) \\ sin\left(\frac{\pi}{4}\right) \end{bmatrix}^\top$

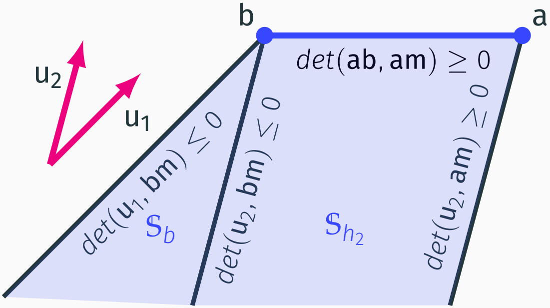
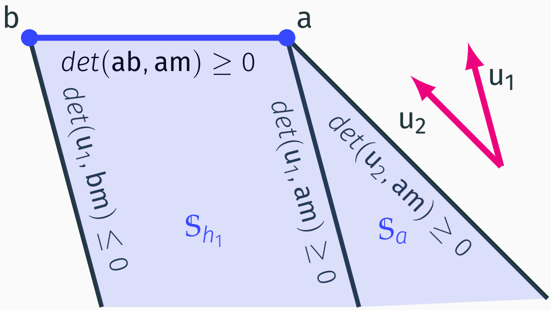


Figure 11: $\langle \mathbf{u}_1, \mathbf{ab} \rangle < 0 \wedge \langle \mathbf{u}_2, \mathbf{ab} \rangle < 0$



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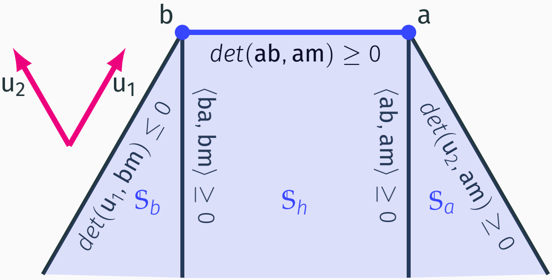

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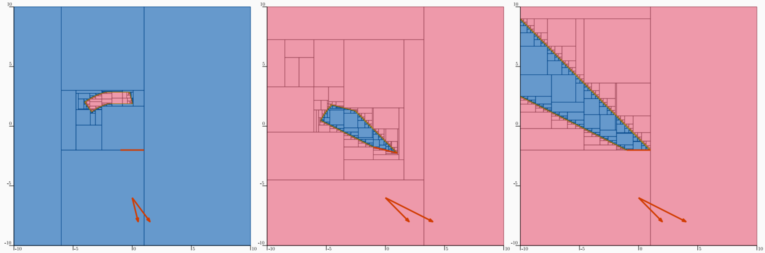
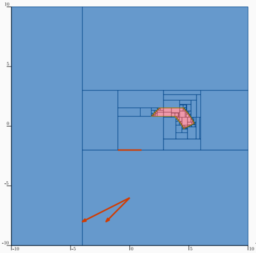


(a) Separator for the remoteness constraint with $d = [4, 5]$

(b) Separator for the remoteness constraint with $d = [5, \infty]$

(c) Separator for the remoteness constraint with $d = [+\infty]$

Figure 12: Remoteness constraint with $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$



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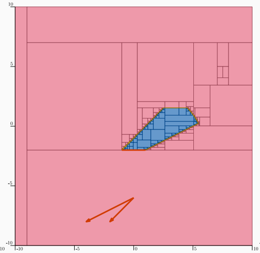
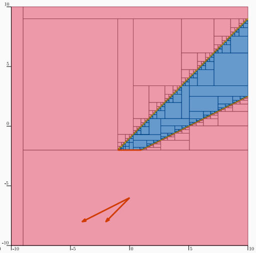
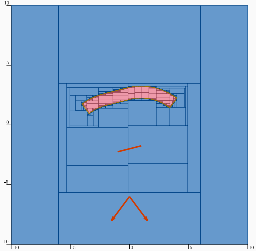


(b) Separator for the remoteness constraint with $d = [5, \infty]$



(c) Separator for the remoteness constraint with $d = [+\infty]$

Figure 13: Remoteness constraint with $\mathbf{u}_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$



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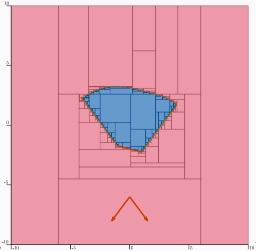

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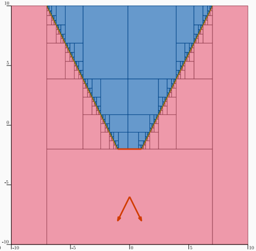


(c) Separator for the remoteness constraint with $d = [+\infty]$

Figure 14: Remoteness constraint with $\mathbf{u}_1 = \begin{bmatrix} -0.75 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0.75 \\ 1 \end{bmatrix}$

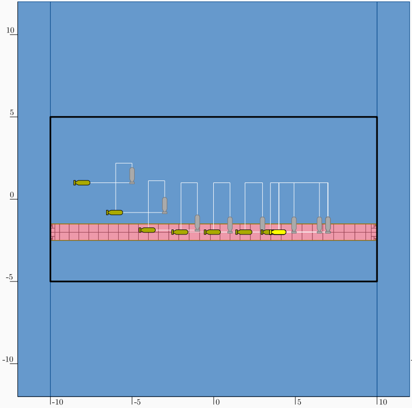
Application

State estimation

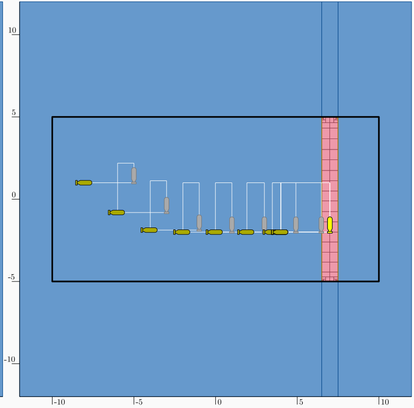
- Robot moving in a 2D plane
- Known map with polygonal obstacles
- Goal: estimate robot's position using remoteness measurements to obstacles

Cycle navigation

- Robot follows a predefined square cyclic path
- Echosounder sense two distances to walls along the cycle



(a) y_0



(b) y_1

Figure 15: Set of position compatible with remoteness measurements

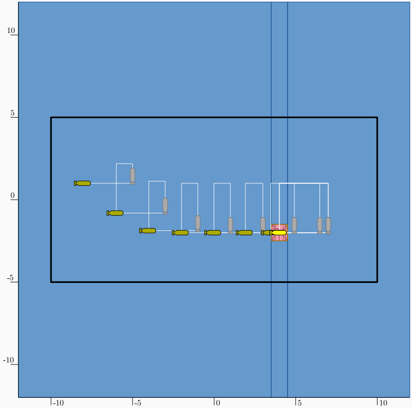


Figure 16: Robot following a square cyclic path in a polygonal environment

Conclusion & Perspectives

Conclusion

- Introduce the χ and the Remoteness constraints
- Implement Separators for both
- Localization of robot in known environment

Perspectives

- Implement online state estimation for the cycle navigation

Questions?

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