On the Robustness of Hyperexponential Stability and Stabilization

Moussa Labbadi
Bretagne INP – iRDL

Séminaire de l'équipe ROBEX
Mercredi 4 novembre 2026
10 h 30 – 12 h

November 04, 2026
Seminar ROBEX

Outline of the talk

1. Introduction
2. Preliminaries on HE
3. Double integrator
   - Control design
   - Differentiator for Signal Estimation
4. Control for a chain of integrators
   - Control in Continuous Time
   - Discretization of the controlled system
5. Illustrative examples
6. Simulations
7. Conclusions and future works

Stabilization at the Origin

A standard control problem: for the plant

$$\dot{x}(t) = f(x(t), u(t))$$

find $u(t) = U(x(t))$ providing stability of the origin with a **prescribed rate**.
Important properties:

1. **Noise sensitivity:** $x(t) \Rightarrow x(t) + v(t)$
2. **Robustness:** $\dot{x}(t) = f\left(x(t), u(t-\tau) + d_1(t), d_2(t)\right)$
   - matched disturbance $d_1(t)$
   - unmatched disturbance $d_2(t)$
   - delay τ
3. **Rate of convergence:**
   - Classical: Asymptotic and Exponential
   - Singular: Finite-time and Fixed-time
   - Accelerated: Hyperexponential
     (faster than any linear system)
4. **Complexity of sampled or discrete-time implementation**

Stability Definitions

Consider the system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad f(0) = 0$$

Denote the solution with initial condition $x_0 \in \mathbb{R}^n$ by

$$X(t, x_0), \qquad 0 \in \Omega \subset \mathbb{R}^n$$

Definition At the equilibrium $x = 0$, the system is said to be:

(a) **Lyapunov stable (LS):** $\forall \varepsilon > 0,\ \exists \delta > 0$ such that $\|x_0\| \leq \delta \quad \Rightarrow \quad \|X(t, x_0)\| \leq \varepsilon, \qquad \forall t \geq 0$

(b) **Asymptotically stable:**
LS and

$$\lim_{t \to +\infty} \|X(t, x_0)\| = 0, \qquad \forall x_0 \in \Omega$$

(c) **Finite-time stable:** LS and $\forall x_0 \in \Omega,\ \exists T_0(x_0) \in \mathbb{R}_+$ such that $X(t, x_0) = 0, \qquad \forall t \geq T_0(x_0)$

(d) **Fixed-time stable:** Finite-time stable and $\sup_{x_0 \in \Omega} T_0(x_0) < +\infty$

(e) **Hyperexponentially stable:** Asymptotically stable and $\lim_{t \to +\infty} \frac{\ln \|X(t, x_0)\|}{t} = -\infty$

The set Ω is called the **domain of stability** or **domain of attraction**.

Uniform Hyperexponential Stability

Consider the following non-autonomous system:

$$\frac{dx(t)}{dt} = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}^+, \tag{1}$$

where:

- $x(t) \in \mathbb{R}^n$ is the state,
- $d(t) \in \mathbb{R}^m$ represents external disturbances, with $d \in L_m^\infty$,
- $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous in x, d, and piecewise continuous in t,
- $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}^+$.

The solution of (1) with initial condition $x_0 \in \mathbb{R}^n$ at $t_0 \in \mathbb{R}^+$ and disturbance $d \in L_m^\infty$ is denoted as:

$$\varphi(t, t_0, x_0, d).$$

Assumption: f ensures the existence of solutions $\varphi(t, t_0, x_0, d)$, at least locally in forward time.
Goal: Introduce a uniform stability property with a hyperexponential rate of convergence.

Definition: Uniform Hyperexponential Stability

Definition **Wang et al. (2024), Wang et al. (2024)**

Given a set $D \subset L_m^\infty$, system (1) is **uniformly hyperexponentially stable** if there exist:

- a constant $\kappa_0 \in \mathbb{R}^+$,
- functions $\rho, \kappa \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$,

such that:

$$\|\varphi(t, t_0, x_0, d)\| \leq \mathrm{e}^{-(\kappa(t-t_0)+\kappa_0)(t-t_0)}\rho(\|x_0\|) + \beta(\|d\|_\infty, t - t_0), \quad \forall t \geq t_0. \tag{2}$$

- Ensures rapid decay of state trajectories.
- The convergence rate is **faster than exponential**.
- Robustness is considered via the disturbance term $\beta(\|d\|_\infty, t - t_0)$.
- Applications in robust control, observer design, and multi-agent systems.

Illustrative Example

Consider the scalar system:

$$\dot{x}(t) = -(1+t)x(t) + d(t), \quad t \geq 0, \tag{3}$$

where $x(t), d(t) \in \mathbb{R}$.
For $t_0 = 0$, the solution satisfies:

$$|x(t)| \leq \mathrm{e}^{-(\frac{t^2}{2}+t)}|x(0)| + \frac{2\|d\|_\infty}{1+(\frac{t^2}{2}+t)}. \tag{4}$$

From the bound, we define:

- $\rho(s) = s$,
- $\kappa(s) = \frac{s}{2}$,
- $\kappa_0 = 1$,
- $\beta(s,t) = \frac{2s}{1+\left(\frac{t}{2}+1\right)t}$.

Key Observations:

- The **hyperexponential rate** applies to the initial conditions.
- Stability is **uniform** in the sense that it is independent of $d \in D$ and t_0.
- While $D \subset L_m^\infty$ is assumed, other input classes can be considered.

Perturbed Double Integrator

$$\begin{aligned} \dot{x}_1 &= x_2 + d_1, \\ \dot{x}_2 &= u + d_2. \end{aligned} \tag{5}$$

where:

- $x = (x_1, x_2)^\top \in \mathbb{R}^2$ is the state vector.
- $u \in \mathbb{R}$ is the control input.
- $d = (d_1, d_2)^\top \in L_2^\infty$ represents bounded perturbations with

$$\|d_i\|_\infty = \sup_{t \geq 0} |d_i(t)| < +\infty, \quad i = 1, 2.$$

Goal: Design a controller handling both perturbations with faster-than-exponential convergence.
Define:

$$\sigma = x_2 + \lambda \psi(t) x_1. \tag{6}$$

where $\lambda > 0$, $\psi(t) = 1 + t$. Ensures hyperexponential convergence.
Any increasing $\psi(t)$ with $\psi(0) > 0$ and an unbounded integral works. Examples: $\psi(t) = a\mathrm{e}^{\alpha t}$, $\psi(t) = bt^t$. Prescribed-time convergence possible via time transformation.

Control Law for Hyperexponential Convergence

To ensure convergence of both x_1 and x_2, we propose the control law:

$$u(t) = -\lambda\left(\psi(t)x_2 + x_1\right) - K\psi(t)^2\sigma(t), \tag{7}$$

where:

- $\lambda > 0$ is a design parameter.
- $K > 1.5\lambda$ is a control gain.
- $\psi(t)$ is a time-varying function ensuring accelerated convergence.

Theorem 1

Under Assumption 1, the double integrator (5) *with control law* (7) *ensures that the closed-loop system is:*

- ***Input-to-State Stable (ISS)** in the presence of disturbances.*
- ***Uniformly Hyperexponentially Stable** in the variable* x_1.

1. Evolution of $\sigma(t)$:

$$\dot{\sigma}(t) = -K\psi(t)^2\sigma(t) + d_2(t) + \lambda\psi(t)d_1(t). \tag{8}$$

Using the time transformation $\tau = Kt\left(\frac{t^2}{3} + t + 1\right)$, we obtain:

$$\frac{d\sigma(\tau)}{d\tau} = -\sigma(\tau) + \frac{K^{-1}d_2(\tau)}{\psi(\tau)^2} + \frac{\lambda K^{-1}d_1(\tau)}{\psi(\tau)}. \tag{9}$$

The bounded solution satisfies:

$$|\sigma(t)| \leq \mathrm{e}^{-Kt\left(\frac{t^2}{3}+t+1\right)}|\sigma(0)| + \frac{r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\psi(t)^2} + \lambda\frac{r_{\frac{3}{K},\frac{1}{3}}K^{-1}\|d_1\|_\infty}{\psi(t)}. \tag{10}$$

2. Evolution of $x_1(t)$:

$$\dot{x}_1 = -\lambda\psi(t)x_1(t) + d_1(t) + \sigma(t). \tag{11}$$

Using the time transformation $\tau = \lambda\frac{t}{2}(t+2)$, we derive:

$$\frac{dx_1(\tau)}{d\tau} = -x_1(\tau) + \frac{d_1(\tau) + \sigma(\tau)}{\lambda\psi(\tau)}. \tag{12}$$

The bounded solution satisfies:

$$|x_1(\tau)| \leq \mathrm{e}^{-\tau}|x_1(0)| + 3\frac{\mathrm{e}^{-\tau} - \mathrm{e}^{-\frac{2K}{3\lambda}\tau}}{2K - 3\lambda}|\sigma(0)| + \frac{r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\lambda\psi(\tau)^3} + \frac{r_{\frac{3}{K},\frac{1}{3}}K^{-1}\|d_1\|_\infty}{\psi(\tau)^2}. \tag{13}$$

Solutions in the Original Time:

$$
\begin{aligned}
|x_1(t)| \leq \mathrm{e}^{-\lambda\frac{t}{2}(t+2)}|x_1(0)| + 3\frac{\mathrm{e}^{-\lambda\frac{t}{2}(t+2)} - \mathrm{e}^{-\frac{K}{3}t(t+2)}}{2K-3\lambda}|\sigma(0)| \\
+ \left(\frac{r_{\frac{2}{\lambda},\frac{1}{2}}}{\lambda\psi(t)} + \frac{r_{\frac{2}{\lambda},1}r_{\frac{3}{K},\frac{1}{3}}K^{-1}}{\psi(t)^2}\right)\|d_1\|_\infty + \frac{r_{\frac{2}{\lambda},\frac{3}{2}}r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\lambda\psi(t)^{\frac{3}{2}}}.
\end{aligned} \tag{14}
$$

This confirms uniform hyperexponential convergence of $\sigma(t), x_1(t)$.

$$
|x_2(t)| = |\sigma(t) - \lambda\psi(t)x_1(t)| \tag{15}
$$

This shows that $x_2(t)$ remains bounded and asymptotically approaches $-d_1(t)$ as $t \to +\infty$ if d_1 is absolutely continuous.

From Theorem 1, $x_2(t)$ converges to $-d_1(t)$ with a hyperexponential decay, leading to a new estimator for the mismatched disturbance, eliminating the need for a disturbance observer.

Alternative control law

Consider the following simple controller:

$$\begin{aligned} u(t) &= -\psi(t)x_2(t) - x_1(t) - \psi(t)\sigma(t) \\ &= -\left(1 + \psi(t)^2\right) x_1(t) - 2\psi(t)x_2(t), \end{aligned} \tag{16}$$

with $\psi(t) = t + \lambda$, $\forall \lambda > 0$.

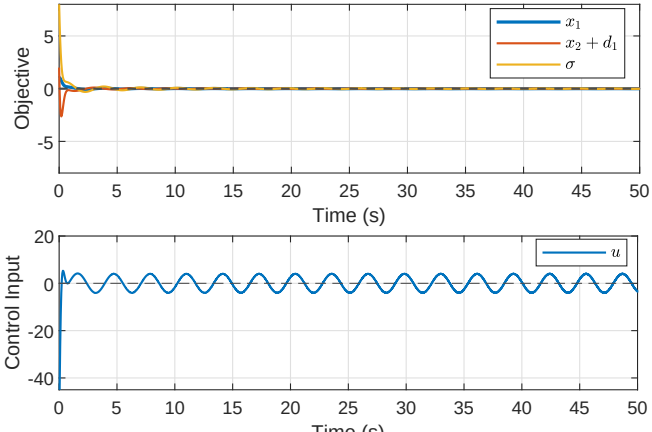
Theorem 2

Under the assumption that $d_i \in \mathcal{L}_\infty$, consider the double integrator (5) *controlled by the algorithm* (16). *Then, the closed-loop system is ISS and uniformly hyperexponentially stable in the variable x_1.*

The difference between the control laws (7) and (16) is that the former guarantees convergence of σ to zero, while the latter does not provide this guarantee.
However, both control laws exhibit similar qualitative behavior for the state dynamics.

Simulation Results

Example 1: Consider the double integrator system (5) with the time-varying controller (7). Let the initial conditions be $x(0) = [2\ 3]^\top$, with the perturbations given by $d_1(t) = \sin(2t)$ and $d_2(t) = 2\cos(2t) + 0.02 \cdot \text{rnd}(1)$, where $\text{rnd}(1)$ denotes random noise. The control gains are set as $\lambda = 4$ and $K = 7$.



Time-Varying Differentiator for Signal Estimation

Consider a differentiable signal $f(t)$ defined on the domain $[0, \infty)$, with a bounded time derivative, i.e., $\|\dot{f}\|_\infty < +\infty$. The objective is to estimate the time derivative of the signal $f(t)$.
To achieve this, we employ the following time-varying differentiator:

$$\dot{\hat{x}}_1 = \hat{x}_2, \quad \dot{\hat{x}}_2 = u(f, \hat{x}),$$

where $u(f, \hat{x})$ is defined in (7) or (16), with the substitutions $\hat{x}_1 - f \to x_1$ and $\hat{x}_2 \to x_2$.
The tracking error is given by:

$$e(t) = \hat{x}_1(t) - f(t),$$

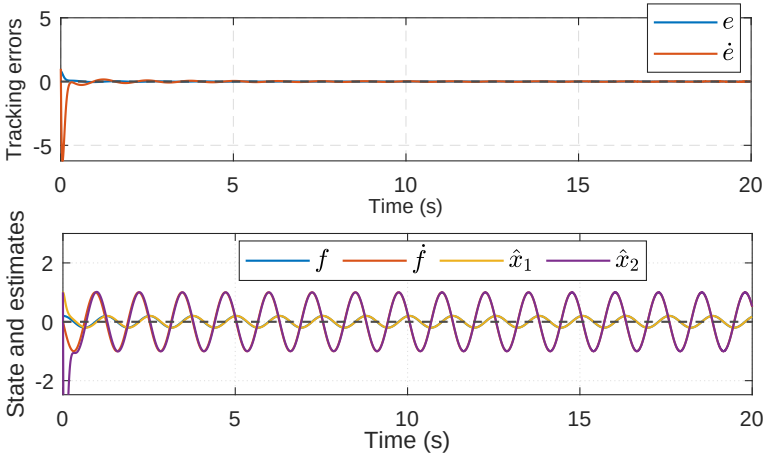
and its derivative is:

$$\dot{e}(t) = \hat{x}_2(t) - \dot{f}(t),$$

which shows that $\hat{x}_2(t)$ can estimate the time derivative $\dot{f}(t)$. This leads to the system (5) with $d_1(t) = -\dot{f}(t)$.

Time-Varying Differentiator for Signal Estimation

The signal is chosen as $f(t) = \frac{1}{5}\cos(5t)$, and the control parameters are $\lambda = 10$ and $K = 20$ for (7), satisfying Theorem 1. The simulation results with initial conditions $\hat{x}(0) = [1, 1]^\top$ are shown in Figure. After an initial reaching phase, $\hat{x}_1(t)$ converges to $f(t)$, and $\hat{x}_2(t)$ converges to $\dot{f}(t)$.



Chain of integrators

Consider a linear system canonical form:

$$\dot{x}(t) = e_n x(t) + d(t) + b_n u(t), \quad t \geq 0, \tag{17}$$

where

$$b_n = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \quad e_n = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix},$$

and $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, and $d(t) = [d_1(t), \ldots, d_n(t)]^\top \in \mathbb{R}^n$ represents the matched and mismatched uncertainties.

The objective is to design a recursive state feedback control law $u(t) = u(x(t), t)$ for this linear system that ensures stabilization of $x(t)$ at the origin with a hyperexponential convergence rate, uniformly in $d \in D = L_\infty^n$.

Control in Continuous Time

Let the auxiliary variables of the control law for (17) be chosen recursively in a linear time-varying form:

$$\begin{aligned}
\sigma_1(t) &= x_1(t), \\
\sigma_2(t) &= \omega_1(t) + \lambda_1 \sigma_1(t)\psi(t), \\
\sigma_3(t) &= \omega_2(t) + \lambda_2 \sigma_2(t)\psi^2(t), \\
&\vdots \\
\sigma_n(t) &= \omega_{n-1}(t) + \lambda_{n-1}\sigma_{n-1}(t)\psi^{n-1}(t),
\end{aligned} \tag{18}$$

where $\lambda_i, i = 1, \ldots, n-1$ are tuning gains such that:

- $\lambda_{i+1} > \lambda_i$,
- $\psi(t) = 1 + t$,
- each recursive term $\sigma_i(t)$ is constructed to reject unmatched perturbations by progressively increasing the gains.

Control in Continuous Time

The term $\omega_i(t)$ represents the dynamics of each $\sigma_i(t)$ such that $\omega_i(t) = \dot{\sigma}_i(t)$ under the condition $d \equiv 0$. For instance, we have:

- $\omega_1(t) = x_2(t)$ because $\dot{\sigma}_1(t) = \dot{x}_1(t) = x_2(t) + d_1(t)$, where $d_1 \equiv 0$ is considered for the design of auxiliary variables.
- Similarly, $\omega_2(t) = x_3(t) + \lambda_1 \left(x_1(t) + x_2(t)\psi(t)\right)$ follows from $\dot{\sigma}_2(t) = x_3(t) + d_2(t) + \lambda_1 \left(x_1(t) + (x_2(t) + d_1(t))\psi(t)\right)$, given that $d_1 = d_2 \equiv 0$.
- We can use the same steps to determine the expressions for all $\omega_i(t)$.

We introduce the desired control law for (17) as follows:

$$u(t) = -\Omega(t, x(t)) - \lambda_n \sigma_n(t) \psi^m(t), \tag{19}$$

where $\Omega(t, x(t))$ contains terms to be compensated on the last step. Here, $\lambda_n > \lambda_{n-1}$ and $m \geq n$.

Control in Continuous Time

Theorem 3

If the tuning gain λ_{i+1} is sufficiently higher than λ_i, under the assumption $d_i \in \mathcal{L}_\infty^n$, consider the linear system (17) controlled by the time-varying algorithm (19). Then, the closed-loop system (17)–(19) is uniformly hyperexponentially stable in the variable x_1 and the variable x_2. If the disturbances d_i, $i = 1, \ldots, n-1$ are $n-i$ times continuously differentiable, then the closed-loop system is ISS.

Corollary 4

For any $\epsilon > 0$ and $D > 0$ such that $\|d\|_\infty \leq D$, there is $\Psi_{\epsilon,D} > 0$ such that by replacing in (18), (19) the gain $\psi(t)$ by $\min\{\psi(t), \Psi_{\epsilon\, D}\}$ we guarantee ISS property of the closed-loop system, while $|x_1(t)| \leq \epsilon$ for some $T_{\epsilon,D}(x(0)) > 0$ for all initial conditions $x(0) \in \mathbb{R}^n$.

Cases Considered in Theorem 3

The theorem considers the following cases:

- If d_i $(i = 1, \ldots, n-1)$ is sufficiently differentiable, a change of coordinates:

$$\tilde{x}_2 = x_2 + d_1, \quad \tilde{x}_3 = x_3 + d_2 + \dot{d}_1, \quad \ldots,$$
$$\tilde{x}_n = x_n + d_{n-1} + \dot{d}_{n-2} + \cdots + \frac{d^{(n-1)}}{dx^{(n-1)}} d_1.$$

  transforms the system into a chain of integrators with matched disturbances. Theorem 3 guarantees uniform hyperexponential stability under sufficient smoothness of d with a common bound on its derivatives.

- If $\psi(t)$ growth saturates when $|x_1| \leq \epsilon$, the asymptotic behavior of x_i $(i = 3, \ldots, n)$ has finite gains in d.

Time Derivative of σ in the Closed-Loop System

The time derivative of σ_n in the closed-loop system is given by:

$$\begin{aligned}\dot{\sigma}_n = &-\lambda_n \psi^m \sigma_n + d_n + P_1^n(\psi) d_{n-1} + P_2^n(\psi) d_{n-2} \\ &+ P_3^n(\psi) d_{n-3} + \cdots + P_{n-2}^n(\psi) d_2 + P_{n-1}^n(\psi) d_1,\end{aligned}$$

where $P_1^n, P_2^n, \ldots, P_{n-1}^n$ are polynomials in ψ of degrees $1, 2, \ldots, n-1$, respectively, with coefficients dependent on λ_i. The same structure applies to the other variables:

$$\begin{aligned}\dot{\sigma}_1 &= -\lambda_1 \psi \sigma_1 + \sigma_2 + d_1, \\ \dot{\sigma}_2 &= -\lambda_2 \psi^2 \sigma_2 + \sigma_3 + d_2 + P_1^2(\psi) d_1, \\ \dot{\sigma}_3 &= -\lambda_3 \psi^3 \sigma_3 + \sigma_4 + d_3 + P_1^3(\psi) d_2 + P_2^3(\psi) d_1, \\ &\vdots \\ \dot{\sigma}_{n-2} &= -\lambda_{n-2} \psi^{n-2} \sigma_{n-2} + \sigma_{n-1} + d_{n-2} \\ &+ P_1^{n-2}(\psi) d_{n-3} + P_2^{n-2}(\psi) d_{n-4} + \cdots + P_{n-3}^{n-2}(\psi) d_1, \\ \dot{\sigma}_{n-1} &= -\lambda_{n-1} \psi^{n-1} \sigma_{n-1} + \sigma_n + d_{n-1} \\ &+ P_1^{n-1}(\psi) d_{n-2} + P_2^{n-1}(\psi) d_{n-3} + \cdots + P_{n-2}^{n-1}(\psi) d_1.\end{aligned}$$

Let us introduce a new time variable as:

$$d\tau = \lambda_n \psi^m(t)\, dt \implies$$

$$\tau = \varphi(t) = \frac{\lambda_n}{m+1}\left(b_1 t + b_2 t^2 + \cdots + b_{m+1} t^{m+1}\right),$$

$$t = \varphi^{-1}(\tau) = \sqrt[m+1]{(m+1)\lambda_n^{-1}\tau + 1} - 1,$$

and $\quad \psi(\tau) = \sqrt[m+1]{(m+1)\lambda_n^{-1}\tau + 1}.$

where b_j are positive coefficients determined based on m according to specific known identities.

The solution is:

$$
\begin{aligned}
|\sigma_n(t)| \leq \mathrm{e}^{-\frac{\lambda_n}{m+1}\left(b_1 t + b_2 t^2 + \cdots + t^{m+1}\right)} |\sigma_n(0)| + \\
\frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m}{m+1}} \|d_n\|_\infty}{\psi^m(t)} + \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m}{m+1}} a_{10} \|d_{n-1}\|_\infty}{\psi^m(t)} \\
+ \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m-1}{m+1}} a_{11} \|d_{n-1}\|_\infty}{\psi^{m-1}(t)} + \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m}{m+1}} a_{20} \|d_{n-2}\|_\infty}{\psi^m(t)} \\
+ \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m-1}{m+1}} a_{21} \|d_{n-2}\|_\infty}{\psi^{m-1}(t)} + \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m-2}, \frac{m}{m+1}} a_{22} \|d_{n-2}\|_\infty}{\psi^{m-2}(t)} \\
+ \dots \\
+ \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m}{m+1}} a_{(n-1)0} \|d_1\|_\infty}{\psi^m(t)} + \dots \\
+ \frac{\lambda_n^{-1} r_{\frac{\lambda_n}{m}, \frac{m-n+1}{m+1}} a_{(n-1)(n-1)} \|d_1\|_\infty}{\psi^{m-n+1}(t)}.
\end{aligned}
$$

with $m > n-1$, this condition imposes an upper bound on σ_n

For the other variables $\sigma_{n-1}, \ldots, \sigma_2$, we can use the same time scale transformations as follows:

$$d\tau = \lambda_{n-1}\psi^{n-1}(t)\,dt, \quad \text{for } \sigma_{n-1}(t),$$
$$\ldots, \quad \text{for } \ldots$$
$$d\tau = \lambda_2\psi^2(t)\,dt, \quad \text{for } \sigma_2(t).$$

After showing the boundedness of all auxiliary variables, we can proceed to the state variables. We begin with the equation

$$\dot{x}_1 = -\lambda_1 x_1 \psi + d_1 + \sigma_2,$$

and apply the following time scale transformation:

$$d\tau = \lambda_1\psi(t)\,dt.$$

This results in the equation for x_1, expressed in terms of the auxiliary variable σ_2, and similarly for x_2, and so on, with x_n obtained from σ_n. We can apply the same process for all variables. As claimed in Theorem 3, the variables x_i with $i = 3, \ldots, n$, their asymptotic behavior is proportional to $d_{i-2}, \ldots, d_1$ with the respective gains $\psi, \ldots, \psi^{i-2}$.

- If the growth of $\psi(t)$ is saturated, as given in Corollary 4, this implies the boundedness of these variables. Furthermore, the asymptotic bound of these variables is proportional to the mismatched d_i, with a gain $r_{a,\alpha}$, and exhibits a hyperexponential decay in the initial conditions vanish.
- If the mismatched d_i are sufficiently smooth, then due to the boundedness and continuity of all variables, and the convergence of $x_1(t)$ to zero, it follows that

$$\dot{x}_1(t) \to 0 \quad \text{as} \quad t \to +\infty,$$

that implies convergence of $x_2(t)$ to $-d_1(t)$. Hence, repeating the same arguments as $t \to +\infty$:

$$x_2(t) = -d_1(t),\ x_3(t) = -d_2(t) - \frac{d}{dt}d_1(t),\ \cdots,$$
$$x_n(t) = -d_{n-1}(t) - \frac{d}{dt}d_{n-2}(t) - \cdots - \frac{d^{n-1}}{dt^{n-1}}d_1(t).$$

Key Points of the Discretization Approach

- **System Structure:** The control system is time-varying with an external input $d(t)$.
- **Instability of Explicit Euler:** Due to increasing $\psi(t)$, explicit Euler discretization becomes unsuitable.
- **Implicit Euler Solution:** Implicit Euler ensures stability and has been used in related works Brogliato et all (2021), Butcher et al.(2008), Wang et al. (2024).
- **Computational Complexity:** Our system depends on $\omega(x)$ and $\Omega(x, \psi)$, introducing challenges.
- **Transformation Strategy:** To simplify discretization, we work in σ-coordinates before mapping back to x-coordinates.

The transformation of the system equations into σ-coordinates has the following form:

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \sigma_2(t) \\ \vdots \\ \sigma_n(t) \end{pmatrix} = S(t)x(t),$$

where $S(t)$ takes the following form:

$$S(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ P_{12}^{\beta_1}(\psi) & 1 & \cdots & 0 & 0 \\ \vdots & P_{13}^{\alpha_1}(\psi) & \ddots & \vdots & \vdots \\ P_{1n-1}^{\beta_{n-1}}(\psi) & \cdots & \cdots & 1 & 0 \\ P_{1n}^{\beta_n}(\psi) & P_{2n}^{\alpha_{2n-2}}(\psi) & \cdots & P_{(n-1)n}^{\gamma}(\psi) & 1 \end{bmatrix}.$$

where $P_{ji}^{\beta_l}(\psi), P_{ji}^{\alpha_l}(\psi), \ldots, P_{ji}^{\gamma}(\psi)$, $I, j = 1, \ldots, n$, are polynomials of $\psi(t)$, and their respective degrees are given as β_l, α_l, and γ. These degrees satisfy the following conditions:

$$\beta_l < \beta_{l+1}, \ \cdots, \ \alpha_l < \alpha_{l+1}, \text{ and } \quad \beta_n > \alpha_{n-2} > \cdots > \gamma.$$

The inverse transformation back to $x(t)$-coordinates is then given by:

$$x(t) = S(t)^{-1}\sigma(t).$$

Consider the dynamics of σ as:

$$\dot{\sigma}(t) = M(t)\sigma(t) + L(t)d(t) \tag{20}$$

$$M(t) = \begin{bmatrix} -\psi(t) & 1 & 0 & \cdots & 0 \\ 0 & -\psi(t)^2 & 1 & \cdots & 0 \\ 0 & 0 & -\psi(t)^3 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -\psi(t)^n \end{bmatrix},$$

and

$$L(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ P_{12}^{\mu_1}(\psi) & 1 & \cdots & 0 & 0 \\ \vdots & P_{23}^{\eta_1}(\psi) & \ddots & \vdots & \vdots \\ P_{1n-1}^{\mu_{n-1}}(\psi) & \cdots & \cdots & 1 & 0 \\ P_{1n}^{\mu_n}(\psi) & P_{2n}^{\eta_{n-2}}(\psi) & \cdots & P_{(n-1)n}^{\nu}(\psi) & 1 \end{bmatrix}.$$

where $P_{ji}^{\mu_j}(\psi), P_{ji}^{\eta_l}(\psi), \ldots, P_{ji}^{\nu}(\psi)$, $I, j = 1, \ldots, n$, are polynomials of $\psi(t)$, and their respective degrees are given as μ_l, η_l, and ν. These degrees satisfy the following conditions:

$$\mu_l < \mu_{l+1}, \cdots, \quad \eta_l < \eta_{l+1}, \quad \text{and} \quad \mu_n > \eta_{n-2} > \cdots > \nu.$$

Let $h > 0$ denote a constant discretization step, and define the discretization time instants by $t_k = hk$, where $k \in \mathbb{Z}^+$. Applying the implicit Euler discretization method to the system equation (20) results in the following update for $k \in \mathbb{Z}_+$:

$$\zeta_{k+1} = Z(t_{k+1}) \left(\zeta_k + hL(t_{k+1}) d_{k+1} \right),$$

where $\zeta_k \in \mathbb{R}^n$ is an approximation of $\sigma_k = \sigma(t_k)$, meaning that $\zeta_k \to \sigma_k$ as $h \to 0$. The sequences $d_k = d(t_k)$ represent the disturbance at the discretization time t_k. The $Z(t)$ is defined as:

$$Z(\rho) = \left[I_n - hM(t) \right]^{-1} \implies Z(\rho) =$$

$$\begin{bmatrix} \frac{1}{\rho_1} & \frac{h}{\rho_1\rho_2} & \frac{h^2}{\rho_1\rho_2\rho_3} & \cdots & \frac{h^{n-1}}{\rho_1\rho_2\cdots\rho_n} \\ 0 & \frac{1}{\rho_2} & \frac{h}{\rho_2\rho_3} & \cdots & \frac{h^{n-2}}{\rho_2\rho_3\cdots\rho_n} \\ 0 & 0 & \frac{1}{\rho_3} & \cdots & \frac{h^{n-3}}{\rho_3\rho_4\cdots\rho_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\rho_n} \end{bmatrix}.$$

$$\rho_i = 1 + h\psi(t)^i, \quad i = 1, 2, \ldots, n.$$

Using $L(t)$, we have:

$$Z(\rho) \times L(t) = \begin{bmatrix} \frac{1}{\rho_1} & \frac{h}{\rho_1\rho_2} & \frac{h^2}{\rho_1\rho_2\rho_3} & \cdots & \frac{h^{n-1}}{\rho_1\rho_2\cdots\rho_n} \\ 0 & \frac{1}{\rho_2} & \frac{h}{\rho_2\rho_3} & \cdots & \frac{h^{n-2}}{\rho_2\rho_3\cdots\rho_n} \\ 0 & 0 & \frac{1}{\rho_3} & \cdots & \frac{h^{n-3}}{\rho_3\rho_4\cdots\rho_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\rho_n} \end{bmatrix}.$$

which allows us to deduce that $\lim_{t\to\infty}(Z(t)L(t)) = 0$.
For this analysis, we assume that the sequence d_k is bounded, i.e., $d_k \in \text{Ł}_n^\infty$, meaning that the disturbance takes finite values with bounded norms.
Now, returning to the coordinate x-system, the closed-loop system is given by the following Implicit Euler discretization:

$$\xi_{k+1} = S(t_{k+1})^{-1} Z(t_{k+1}) \left(S(t_{k+1})\xi_k + hL(t_{k+1})d_{k+1}\right), \tag{21}$$

where $\xi_k \in \mathbb{R}^n$ is an approximation of $x_k = x(t_k)$, and it holds that $\xi_k \to x_k$ as $h \to 0$.

Theorem 5

Let the auxiliary variable σ be chosen such that $\beta_l < \beta_{l+1}, \quad \cdots, \quad \alpha_l < \alpha_{l+1}, \quad$ and $\quad \beta_n > \alpha_{n-2} > \cdots > \gamma, \quad \mu_l < \mu_{l+1}, \quad \cdots, \quad \eta_l < \eta_{l+1}, \quad$ and $\quad \mu_n > \eta_{n-2} > \cdots > \nu$. Then, the system (21) exhibits ISS. Moreover, the convergence to the origin is faster than any exponential.

We examine the following limits:

$$\lim_{t\to\infty} \left(S(t)^{-1} Z(t) S(t)\right) = -\frac{1}{h^{n-1}} \begin{bmatrix} 0 & 0 & \cdots & 0 & \\ h^{n-2} & 0 & \cdots & 0 & \\ h^{n-3} & h^{n-2} & 0 & \cdots & \\ \vdots & h^{n-3} & h^{n-4} & \cdots & 0 \\ 1 & h & h^2 & \cdots & h^{n-2} \end{bmatrix},$$

which implies the hyperexponential stability of the system (21).

$$\lim_{t\to\infty} \left(S(t)^{-1} Z(t) L(t)\right) = -\frac{1}{h^{n-1}} \begin{bmatrix} 0 & 0 & \cdots & 0 & \\ h^{n-2} & 0 & \cdots & 0 & \\ h^{n-3} & h^{n-2} & 0 & \cdots & \\ \vdots & h^{n-3} & h^{n-4} & \cdots & 0 \\ 1 & h & h^2 & \cdots & h^{n-2} \end{bmatrix}.$$

Example 1

Let us consider a chain of integrators for $n = 3$. We define auxiliary variables to facilitate the analysis of the system dynamics:

$$\begin{aligned}\sigma_1(t) &= x_1(t),\\ \sigma_2(t) &= x_2(t) + \psi(t)x_1(t),\\ \sigma_3(t) &= x_3(t) + \psi(t)x_2(t) + x_1(t) + \psi(t)^2\sigma_2(t),\end{aligned}$$

where $\psi(t) = 1 + t$.
The control law is constructed as:

$$\begin{aligned}u(t) = &-\psi(t)^3\sigma_3(t) - \psi(t)\left(x_3(t) + 2\sigma_2(t)\right) - 2x_2(t)\\ &+ \psi(t)^2\left(\psi(t)^2\sigma_2(t) - \sigma_3(t)\right).\end{aligned} \tag{22}$$

Corollary 6

Under the controller (22) *and the discretization in Theorem 5, the system* (21) *for* $n = 3$ *exhibits ISS. Furthermore, the system converges to the origin faster than exponentially.*

We examine the dynamics of the auxiliary variables:

$$\dot{\sigma}_1(t) = -\psi(t)\sigma_1(t) + d_1(t) + \sigma_2(t),$$
$$\dot{\sigma}_2(t) = -\psi(t)^2\sigma_2(t) + d_2(t) + \psi(t)d_1(t) + \sigma_3(t),$$
$$\dot{\sigma}_3(t) = -\psi(t)^3\sigma_3(t) + \psi(t)(1+\psi(t))d_2(t) + (1+\psi(t)^3)d_1(t) + d_3(t).$$

Next, we define the coordinate transformation:

$$\sigma(t) = \begin{pmatrix}\sigma_1(t)\\ \sigma_2(t)\\ \sigma_3(t)\end{pmatrix} = S(t)x(t), \quad S(t) = \begin{bmatrix}1 & 0 & 0\\ P_{12}^1(\psi) & 1 & 0\\ P_{13}^3(\psi) & P_{13}^2(\psi) & 1\end{bmatrix},$$

where: $P_{12}^1(\psi) = \psi(t),\ P_{13}^3(\psi) = \psi(t)^3 + 1,\ P_{23}^2(\psi) = (\psi(t)+1)\psi(t)$. The inverse matrix $S(t)^{-1}$ is:

$$S(t)^{-1} = \begin{bmatrix}1 & 0 & 0\\ P_{12}^1(\psi) & 1 & 0\\ P_{13}^2(\psi) & P_{23}^2(\psi) & 1\end{bmatrix},$$

where: $P_{12}^1(\psi) = -\psi(t),\ P_{13}^2(\psi) = \psi(t)^2 - 1), P_{23}^2(\psi) = -\psi(t)^2 - \psi(t))$.
Since $\det(S(t)) = 1$, the transformation is invertible, preserving the structure of the system.
By applying the chain rule, we derive the dynamics of the transformed variables $\sigma(t)$:

$$\dot{\sigma}(t) = M(t)\sigma(t) + L(t)d(t),$$

where $M(t)$ and $L(t)$ are time-varying matrices defined as:

$$M(t) = \begin{bmatrix} -\psi(t) & 1 & 0 \\ 0 & -\psi(t)^2 & 1 \\ 0 & 0 & -\psi(t)^3 \end{bmatrix}, L(t) = \begin{bmatrix} 1 & 0 & 0 \\ P_{12}^1(\psi) & 1 & 0 \\ P_{13}^3(\psi) & P_{23}^2(\psi) & 1 \end{bmatrix}.$$

where $P_{12}^1(\psi) = \psi(t)$, $P_{13}^3 = \psi(t)^3 + 1$, and $P_{23}^2(\psi) = (\psi(t) + 1)\psi(t)$.
To discretize the system, we introduce the following implicit discretization scheme:

$$\zeta_{k+1} = \zeta_k + h\left(M(t_{k+1})\zeta_{k+1} + L(t_{k+1})d_{k+1}\right).$$

where $\zeta_k \in \mathbb{R}^3$ is an approximation of $\sigma_k = \sigma(t_k)$, meaning that $\zeta_k \to \sigma_k$ as $h \to 0$.
Rearranging this, we obtain:

$$\zeta_{k+1} = Z(t_{k+1})\left(\zeta_k + hL(t_{k+1})d_{k+1}\right),$$

where $Z(t)$ is a matrix that depends on $\psi(t)$ and is given by:

$$Z(t) = \begin{bmatrix} \frac{1}{\rho_1} & \frac{h}{\rho_1\rho_2} & \frac{h^2}{\rho_1\rho_2\rho_3} \\ 0 & \frac{1}{\rho_2} & \frac{h}{\rho_2\rho_3} \\ 0 & 0 & \frac{1}{\rho_3} \end{bmatrix}, \quad \rho_i = 1 + h\psi(t)^i, \quad i = 1, 2, 3.$$

Finally, we describe the closed-loop system, which is given by:

$$\xi_{k+1} = S(t_{k+1})^{-1} \left[Z(t_{k+1}) \left(S(t_{k+1})\xi_k + hL(t_{k+1})d_{k+1} \right) \right]. \tag{23}$$

where $\xi_k \in \mathbb{R}^3$ is an approximation of $x_k = x(t_k)$, and it holds that $\xi_k \to x_k$ as $h \to 0$. As $t \to \infty$, the behavior of the system can be characterized by the following limits:

$$\lim_{t\to\infty} \left(S(t)^{-1} Z(t) S(t) \right) = -\frac{1}{h^2} \begin{bmatrix} 0 & 0 & 0 \\ h & 0 & 0 \\ 1 & h & 0 \end{bmatrix},$$

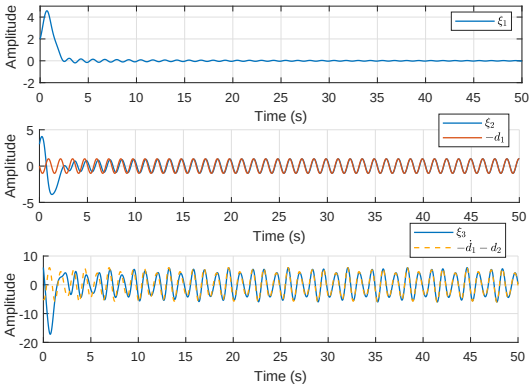
which exhibits a hyperexponential decay. For robustness, consider the following term.

$$\lim_{t\to\infty} \left(S(t)^{-1} Z(t) L(t) \right) = -\frac{1}{h^2} \begin{bmatrix} 0 & 0 & 0 \\ h & 0 & 0 \\ 1 & h & 0 \end{bmatrix}.$$

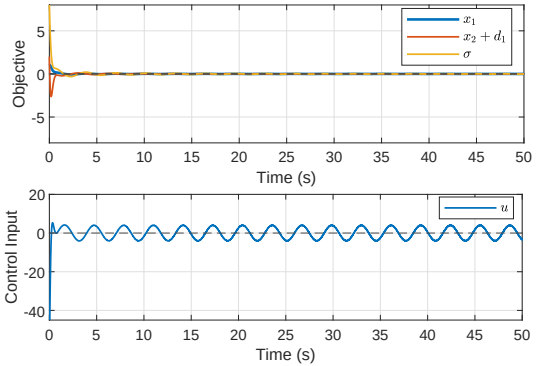
Thus, we conclude that the system exhibits robust stability and faster-than-exponential convergence to the origin.

Simulations for $n = 3$

For $h = 0.001$ and the perturbations $d_1(t) = \sin(5t)$, $d_2(t) = \sin(7t)$, $d_3(t) = \cos(3t) - \mathrm{rnd}(1)$,



(a) Objective vs time.



(b) Error norms in logarithmic scale vs time.

Conclusions and Future Works

Conclusions:

- ⋆ A new control strategy for a chain of integrators subject to unmatched perturbations is proposed.
- ▶ The continuous-time approach ensures **hyperexponential convergence** for x_1, while x_2 remains bounded.
- ✓ For higher-order states $n \geq 3$, **input-to-state stability (ISS)** is achieved via saturated growing gain.
- ◇ A simple discrete-time implementation is introduced, guaranteeing **uniform hyperexponential convergence** despite matched and unmatched inputs.

Future Works:

- ⇒ Extending the approach to **nonlinear systems** with state and input constraints.
- ▷ Expanding the class of **admissible perturbations** to include those **proportional to the state vector**.
- ⟹ Investigating applications to **nonholonomic systems**, particularly **perturbed unicycle-type mobile robots**.

References

- **Wang, J., Zimenko, K., Polyakov, A., Efimov, D. (2024)**: *An exact robust high-order differentiator with hyperexponential convergence.*
  IEEE Transactions on Automatic Control
- **Wang, J., Zimenko, K., Efimov, D., Polyakov, A. (2024)**: *On hyperexponential stabilization of a chain of integrators in continuous and discrete time.*
  Automatica, **166**, 111723
- **Labbadi, M., Efimov, D. (2024)**: *Hyperexponential Stabilization of Double Integrator with Unmatched Perturbations.*
  IEEE Control Systems Letters
- **Efimov, D., Polyakov, A., Zimenko, K., Wang, J. (2022)**: *An exact robust hyperexponential differentiator.*
  2022 IEEE 61st Conference on Decision and Control (CDC), 1894–1899

Thank You for Your Attention!

Questions are welcome!

Merci!

Moussa Labbadi
Bretagne INP - iRDL Laboratory
moussa.labbadi@enib.fr