Validated prediction using intervals and integrals

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1. Reachability

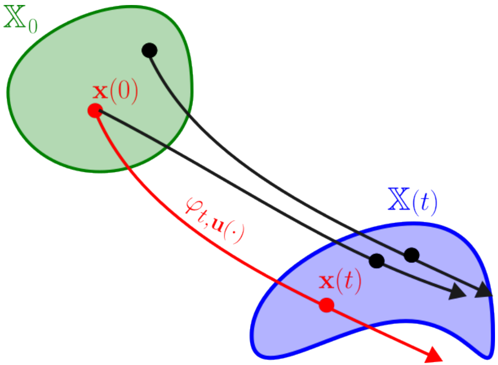
We have

- a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$
- an uncertain input $\mathbf{u}(t) \in [\mathbf{u}]$
- an initial state vector $\mathbf{x}(0) \in \mathbb{X}_0$

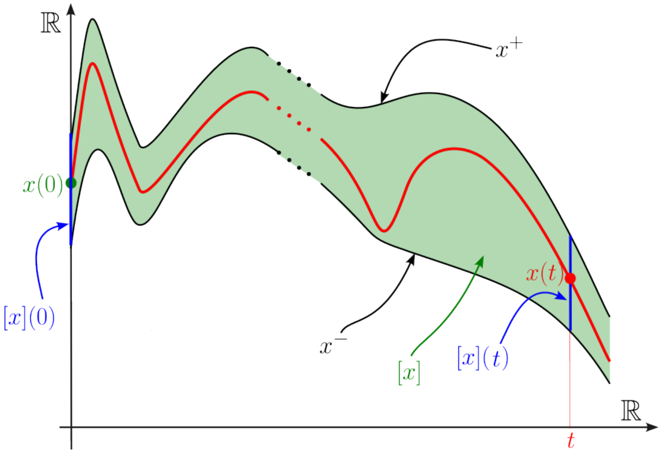
The *reach set is*

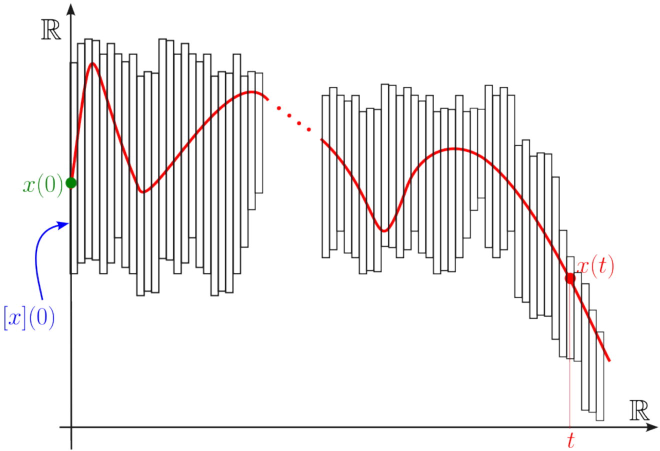
$$\mathbb{X}(t) = \left\{ \mathbf{a} | \, \exists \mathbf{x}(0) \in \mathbb{X}_0, \exists \mathbf{u}(\cdot) \in [\mathbf{u}], \mathbf{a} = \varphi_{t,\mathbf{u}(\cdot)}(\mathbf{x}(0)) \right\}$$

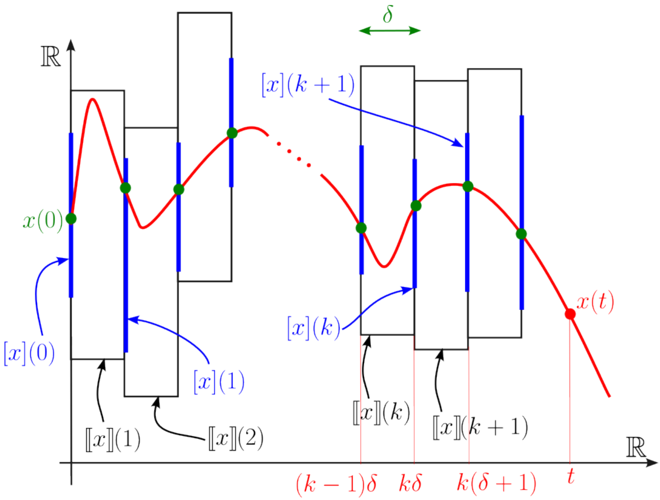
where $\varphi_{t,\mathbf{u}(\cdot)}$ is the flow.


$\mathbb{X}_0$
$\mathbf{x}(0)$
$\varphi_{t,\mathbf{u}(\cdot)}$
$\mathbb{X}(t)$
$\mathbf{x}(t)$

Interval tubes

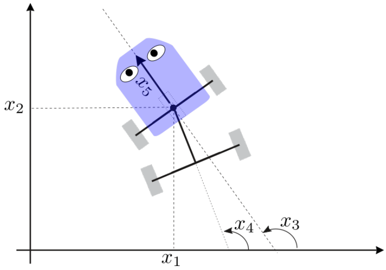

$\mathbb{R}$
x^+
$x(0)$
$[x](0)$
x^-
$[x]$
$[x](t)$
$x(t)$
$\mathbb{R}$
t


ℝ
$x(0)$
$[x](0)$
$x(t)$
ℝ
t


δ
$\mathbb{R}$
$[x](k+1)$
$x(0)$
$[x](0)$
$[x](1)$
$[[x]](1)$
$[[x]](2)$
$[x](k)$
$[[x]](k)$
$[[x]](k+1)$
$x(t)$
$\mathbb{R}$
$(k-1)\delta$
$k\delta$
$k(\delta+1)$
t

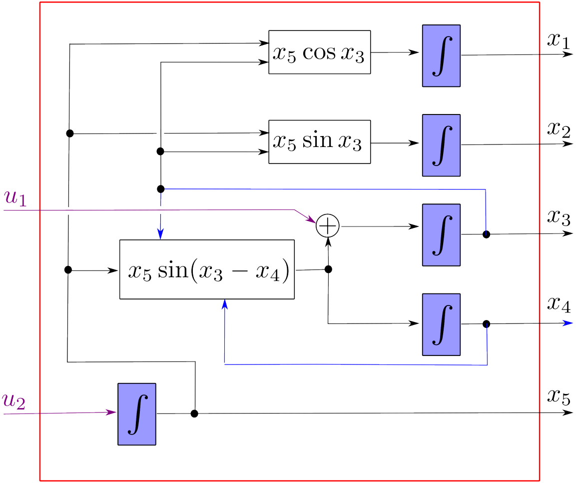
Consider two signals $x(\cdot) \in [x](\cdot)$ and $y(\cdot) \in [y](\cdot)$. We have:

$$\begin{array}{ccc} x(\cdot)+y(\cdot) & \in & [x](\cdot)+[y](\cdot) \\ x(\cdot)*y(\cdot) & \in & [x](\cdot)*[y](\cdot) \\ \sin(x(\cdot)) & \in & \sin([x](\cdot)) \\ \int_0 x(\cdot) & \in & \int_0 [x](\cdot) \\ \vdots & \vdots & \vdots \end{array}$$



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$

Boatbot towing a magnetometer


$x_5 \cos x_3$
$x_5 \sin x_3$
$x_5 \sin(x_3 - x_4)$
x_1
x_2
x_3
x_4
x_5
u_1
u_2

3. Integral algebra

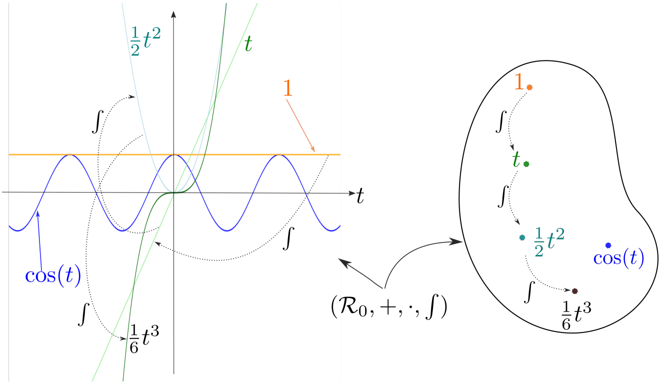
An *integral ring* is a a ring $(\mathscr{R},+,\cdot)$ equipped with the integration $\int$ such that

(i) $\mathbb{R} \subset \mathscr{R}$

(ii) $\forall a \in \mathscr{R}, \int a \in \mathscr{R}$

Consider $\mathscr{R}_0$ the smallest real integral ring.
We have

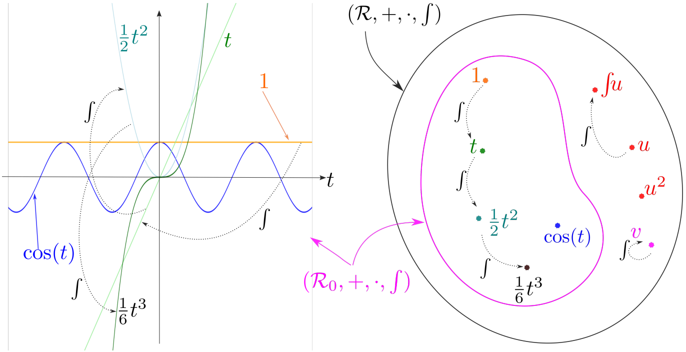
$$\begin{aligned} a = 2 \in \mathscr{R}_0 & \qquad \text{it is a constant} \\ b = 2t \in \mathscr{R}_0 & \qquad \text{since, } b = \textstyle\int a \end{aligned}$$


$\frac{1}{2}t^2$
t
1
$\int$
t
$\int$
$\cos(t)$
$\int$
$\frac{1}{6}t^3$
$(\mathcal{R}_0, +, \cdot, \int)$
1
$\int$
t
$\int$
$\frac{1}{2}t^2$
$\cos(t)$
$\int$
$\frac{1}{6}t^3$

Consider an *integral ring extension* is $\mathscr{L}/\mathscr{R}$.
An element u of $\mathscr{L}$ is said to be integral $\mathscr{R}$-algebraic independent if

$$u, \int u, \int^2 u, \int^3 u, \ldots$$

are all independent.


$\frac{1}{2}t^2$
t
1
$\int$
t
$\int$
$\cos(t)$
$\int$
$\frac{1}{6}t^3$
$(\mathcal{R}, +, \cdot, \int)$
1
$\int$
t
$\int$
$\frac{1}{2}t^2$
$\int$
$\frac{1}{6}t^3$
$\cos(t)$
$\int u$
$\int$
u
u^2
v
$\int$
$(\mathcal{R}_0, +, \cdot, \int)$

Integral dynamical system

Given an integral ring $\mathscr{R}$.
We denote by $\mathscr{R} < u_1, u_2, \cdots >$ the integral ring generated by $\mathscr{R}$ and by a finite set $\{u_1, u_2, \ldots\}$ that are integral $\mathscr{R}$-algebraic independent.

Example. Consider the integral ring $\mathscr{L} = \mathscr{R}_0 < u >$. We have

$$\begin{array}{rcl} \cos t & \in & \mathscr{L} \\ u + \int \sin u + 3 & \in & \mathscr{L} \\ u + \int \left(\sin \int^3 u \right) + 3 & \in & \mathscr{L} \end{array}$$

Definition. Consider a system of the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

This system is an *integral dynamical system* if for all $i \in \{1, \ldots, n\}$, $x_i \in \mathscr{R}_0 < u_1, \ldots, u_m >$.

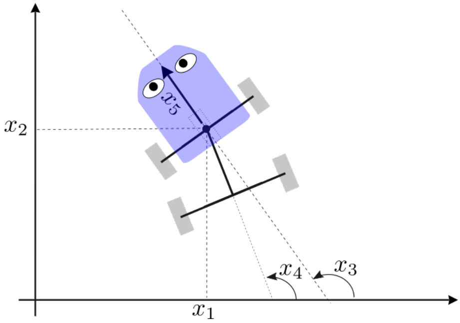
Interval extension of an integral dynamical system

Consider an integral dynamical system $\{x_1 \dots, x_n\} \in \mathscr{R}_0 < u_1, \dots, u_m >$.

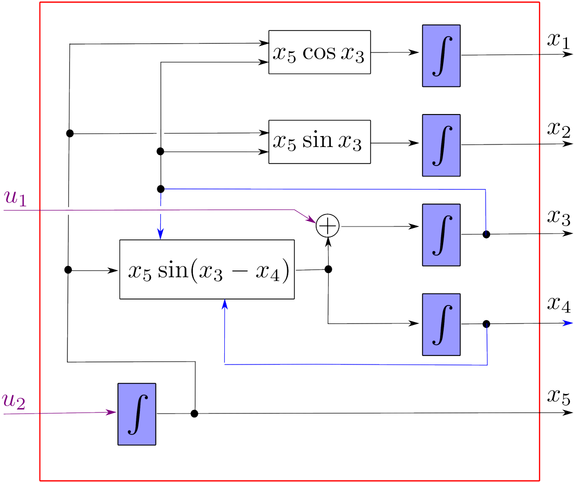
- For each x_i, we can build an expression which involves $x_1(0), \dots, x_n(0)$, $u_1, \dots, u_m$ as variables and $+, -, \cdot, /, \int$ as operators.
- An interval evaluation for the x_i's can be performed using the classical rules of interval arithmetic.

4. Applications

The Car-Trailer



$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \cos x_3 \\ x_5 \sin x_3 \\ u_1 + x_5 \sin(x_3 - x_4) \\ x_5 \sin(x_3 - x_4) \\ u_2 \end{pmatrix}$$



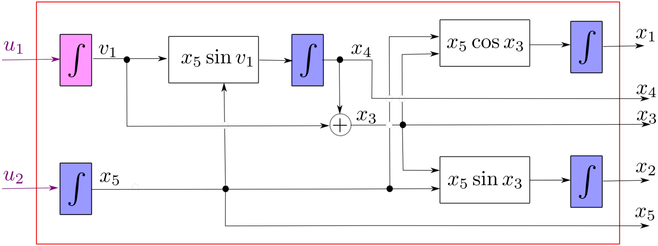
Can we conclude that $x_1, \ldots, x_5$ belong or not to $\mathscr{R}_0 < u_1, u_2 >$?

Proposition. An integral formulation of the car-trailer is

$$\begin{cases} x_1 &= x_1(0) + \int (x_5 \cos x_3) \\ x_2 &= x_2(0) + \int (x_5 \sin x_3) \\ x_3 &= x_4 + v_1 \\ x_4 &= x_4(0) + \int (x_5 \sin v_1) \\ v_1 &= v_1(0) + \int u_1 \\ x_5 &= x_5(0) + \int u_2 \end{cases}$$

with

$$v_1(0) = x_3(0) - x_4(0).$$



Integral representation of the car-trailer

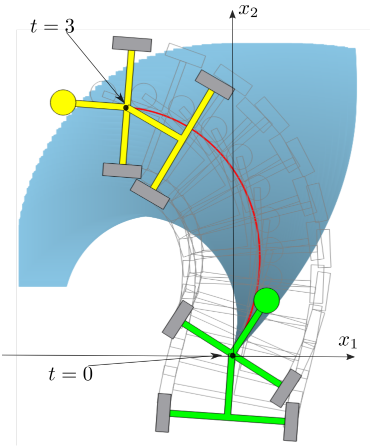
The interval trajectory is obtained by:

$$\begin{array}{rcl}
\text{In:} & & [x_1](0),[x_2](0),[x_3](0),[x_4](0),[x_5](0),[u_1](t),[u_2](t) \\
[v_1](0) & = & [x_3](0)-[x_4](0). \\
[v_1](t) & = & [v_1](0)+\int_0^t [u_1](\tau)d\tau \\
[x_5](t) & = & [x_5](0)+\int_0^t [u_2](\tau)d\tau \\
[x_4](t) & = & [x_4](0)+\int_0^t [x_5](\tau)\cdot\cos([v_1](\tau))\cdot d\tau \\
[x_3](t) & = & [x_4](t)+[v_1](t) \\
[x_1](t) & = & [x_1](0)+\int_0^t [x_5](\tau)\cdot\cos([x_3](\tau))\cdot d\tau \\
[x_2](t) & = & [x_2](0)+\int_0^t [x_5](\tau)\cdot\sin([x_3](\tau))\cdot d\tau
\end{array}$$

Take

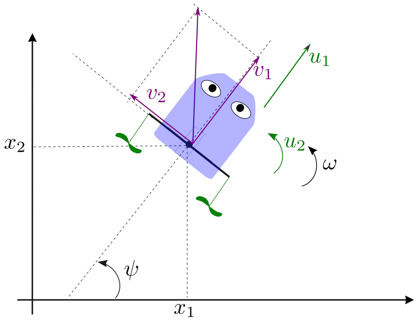
$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \begin{pmatrix} [u_1](t) \\ [u_2](t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + 10^{-2} \begin{pmatrix} [-1,1] \\ [-1,1] \end{pmatrix}$$

$$\mathbf{x}(0) \in [\mathbf{x}](0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1.5 \\ 1 \end{pmatrix} + \begin{pmatrix} [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.2, 0.2] \\ [-0.01, 0.01] \\ [-0.001, 000.1] \end{pmatrix}.$$



Integral simulation of the car-trailer

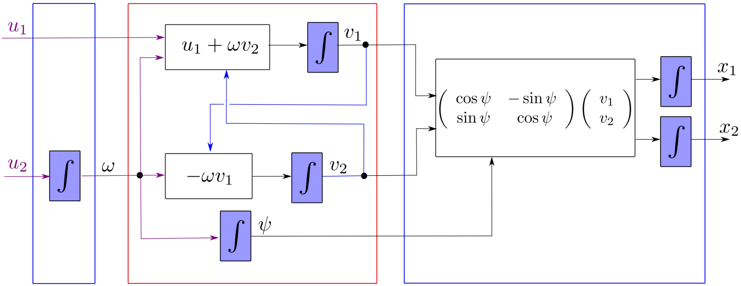
The hovercraft



The hovercraft has two propellers and can glide in all directions without any friction

The state equations are given by

$$\begin{cases} \dot{x}_1 &= v_1 \cos\psi - v_2 \sin\psi \\ \dot{x}_2 &= v_1 \sin\psi + v_2 \cos\psi \\ \dot{v}_1 &= u_1 + \omega v_2 \\ \dot{v}_2 &= -\omega v_1 \\ \dot{\psi} &= \omega \\ \dot{\omega} &= u_2 \end{cases}$$

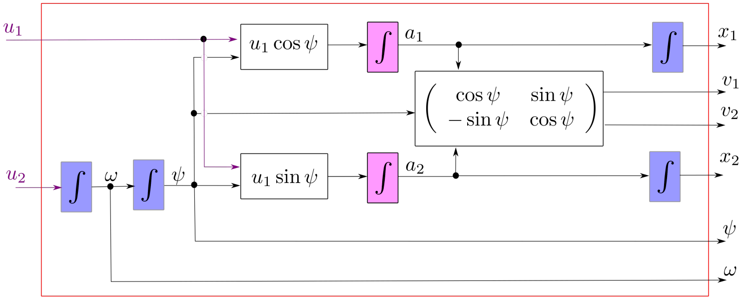

u_1
u_2
$\int$
ω
$u_1 + \omega v_2$
$\int$
v_1
$-\omega v_1$
$\int$
v_2
$\int$
ψ
$\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$
$\int$
x_1
$\int$
x_2

Proposition. An integral formulation of the hovercraft is

$$\left\{\begin{array}{ccc}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & = & \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} \int(\cos\psi \cdot v_1 - \sin\psi \cdot v_2) \\ \int(\sin\psi \cdot v_1 + \cos\psi \cdot v_2) \end{pmatrix} \\
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & = & \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \left(\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} + \begin{pmatrix} \int(u_1\cos\psi) \\ \int(u_1\sin\psi) \end{pmatrix} \right) \\
\psi & = & \psi(0) + \int\omega \\
\omega & = & \omega(0) + \int u_2
\end{array}\right.$$

where

$$\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} \cos\psi(0) & -\sin\psi(0) \\ \sin\psi(0) & \cos\psi(0) \end{pmatrix} \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}.$$


u_1
u_2
$u_1 \cos\psi$
$u_1 \sin\psi$
a_1
a_2
ω
ψ
$\begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}$
x_1
v_1
v_2
x_2
ψ
ω

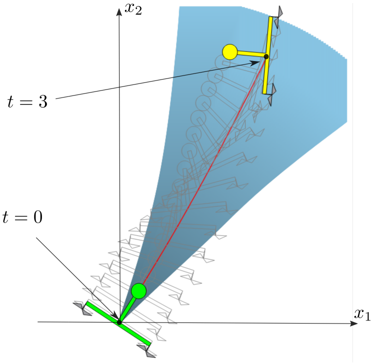
Take

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \begin{pmatrix} [u_1](t) \\ [u_2](t) \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} + \begin{pmatrix} [-0.01, 0.01] \\ [-0.01, 0.01] \end{pmatrix}$$

$$\mathbf{x}(0) \in [\mathbf{x}](0) = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.001, 0.001] \\ [-0.2, 0.2] \\ [-0.001, 0.001] \end{pmatrix}$$

The interval trajectory in the (x_1, x_2)-space is obtained by

$$
\begin{array}{rcl}
\text{In:} & & [\mathbf{x}](0), [\mathbf{v}](0), [\psi](0), [\omega](0), [\mathbf{u}](t) \\
[\mathbf{a}](0) & = & \begin{pmatrix} \cos([\psi](0)) & -\sin([\psi](0)) \\ \sin([\psi](0)) & \cos([\psi](0)) \end{pmatrix} \cdot [\mathbf{v}](0) \\
[\mathbf{a}](t) & = & [\mathbf{a}](0) + \begin{pmatrix} \int_0^t [u_1](\tau) \cdot \cos([\psi](\tau)) \cdot d\tau \\ \int_0^t [u_1](\tau) \cdot \sin([\psi](\tau)) \cdot d\tau \end{pmatrix} \\
[\omega](t) & = & [\omega](0) + \int_0^t [u_2](\tau) d\tau \\
[\psi](t) & = & [\psi](0) + \int_0^t [\omega](\tau) d\tau \\
[\mathbf{v}](t) & = & \begin{pmatrix} \cos([\psi](t)) & \sin([\psi](t)) \\ -\sin([\psi](t)) & \cos([\psi](t)) \end{pmatrix} \cdot [\mathbf{a}](t) \\
[\mathbf{x}](t) & = & [\mathbf{x}](0) + \begin{pmatrix} \cos([\psi](t)) & -\sin([\psi](t)) \\ \sin([\psi](t)) & \cos([\psi](t)) \end{pmatrix} \cdot [\mathbf{v}](t)
\end{array}
$$


x_2
$t = 3$
$t = 0$
x_1

References

1. Integral algebra [4]
2. Validating trajectory [13][3]
3. Tank-Trailer Model [10]
4. Hovercraft [2]
5. Monotone systems [12]
6. Interval analysis [6][5]
7. Tubes: interval tube arithmetic [1][9][11][8]
8. Interval for control [7]

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