A new wrapper for a reliable resolution of underdetermined nonlinear equations

Luc Jaulin

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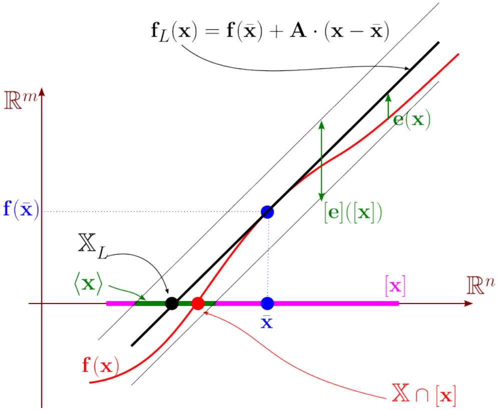
First order enclosure

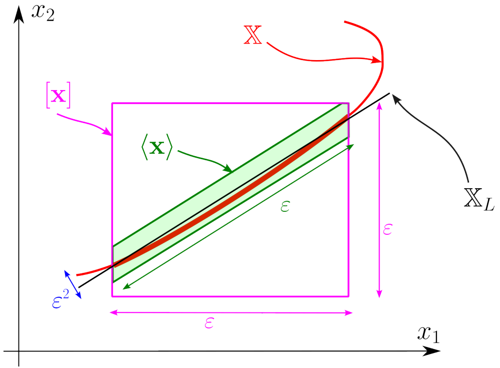
We want to characterize the set

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$, $m < n$.



Principle of the first order enclosure: $w([\mathbf{e}]([\mathbf{x}])) = o(w([\mathbf{x}]))$



$$\mathrm{Vol}(\langle \mathbf{x} \rangle) = O(\varepsilon^n \cdot \varepsilon^{k(n-m)}), k = 1 \text{ whereas } \mathrm{Vol}([\mathbf{x}]) = O(\varepsilon^n)$$

Polyhedron approximation of $\mathbb{X} \cap [\mathbf{x}]$

Proposition

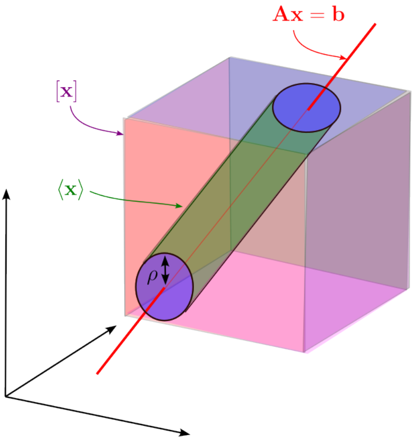
$$\left\{ \begin{array}{c} \mathbf{f}(\mathbf{x}) = \mathbf{0} \\ \mathbf{x} \in [\mathbf{x}] \end{array} \right. \Rightarrow \left\{ \begin{array}{c} \mathbf{A} \cdot \mathbf{x} \in [\mathbf{b}] \\ \mathbf{A} = \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}}) \\ [\mathbf{b}] = \mathbf{A}\bar{\mathbf{x}} - \mathbf{f}(\bar{\mathbf{x}}) + [\mathbf{e}] \\ [\mathbf{e}] = \left(\mathbf{A} - \frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])\right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \end{array} \right.$$

Buches

The *buche* (French name for *log*) associated to $[\mathbf{x}] \subset \mathbb{R}^n$, a matrix $\mathbf{A}$, a vector $\mathbf{b}$ and the radius ρ is

$$\begin{aligned} \langle \mathbf{x} \rangle &= \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle \\ &= \{ \mathbf{x} \in [\mathbf{x}], \exists \mathbf{p}, \mathbf{A}\mathbf{p} = \mathbf{b} \text{ and } \|\mathbf{x} - \mathbf{p}\| < \rho \}. \end{aligned}$$

The affine space $\mathbf{A}\mathbf{p} = \mathbf{b}$ is called a *flat*.


$\mathbf{Ax} = \mathbf{b}$
$[\mathbf{x}]$
$\langle\mathbf{x}\rangle$
ρ

Motivation for using buches is:

- The box $[\mathbf{x}]$ in the buche allows to build a nonoverlapping covering of $\mathbb{X}$.
- A buche can easily be bisected
- The axis-aligned projection is easy
- Buches can easily be intersected
- A first order approximation is possible

Projection of a buche

Consider

$$\mathbb{R}^n = \left\{ \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_1 \in \mathbb{R}^m, \mathbf{x}_2 \in \mathbb{R}^{n-m} \right\}.$$

The projection of $\langle \mathbf{x} \rangle = \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$ on the $\mathbf{x}_1$-space is defined by:

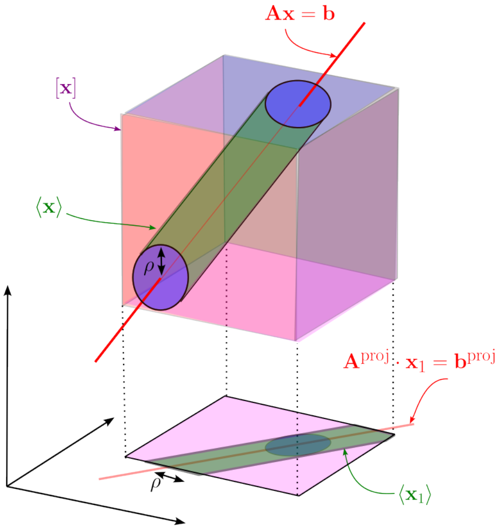
$$\mathsf{proj}_{1:m} \langle \mathbf{x} \rangle = < [\mathbf{x}_1], \mathbf{A}^{\mathsf{proj}}, \mathbf{b}^{\mathsf{proj}}, \rho >$$

where

$$\begin{array}{ccc} (\mathbf{A}_1, \mathbf{A}_2) & = & \mathbf{A} \\ \mathbf{A}^{\mathsf{proj}} & = & \left(\mathbf{A}_1^{-1} \mathbf{A}_2 \right)^{\perp} \\ \mathbf{b}^{\mathsf{proj}} & = & \mathbf{A}^{\mathsf{proj}} \mathbf{A}_1^{-1} \mathbf{b} \\ [\mathbf{x}_1] & = & \mathsf{proj}_{1:m}([\mathbf{x}]) \end{array}$$

Proposition.

$$\mathbf{x} \in \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle \Rightarrow \mathsf{proj}_{1:m}\mathbf{x} \in \mathsf{proj}_{1:m} \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$


$\mathbf{Ax} = \mathbf{b}$
$[\mathbf{x}]$
$\langle \mathbf{x} \rangle$
ρ
$\mathbf{A}^{\mathrm{proj}} \cdot \mathbf{x}_1 = \mathbf{b}^{\mathrm{proj}}$
ρ
$\langle \mathbf{x}_1 \rangle$

Intersection of two buches

The intersection between $\langle \mathbf{x}_1 \rangle$ and $\langle \mathbf{x}_2 \rangle$ is defined by

$$\langle [\mathbf{x}]_1, \mathbf{A}_1, \mathbf{b}_1, \rho_1 \rangle \cap \langle [\mathbf{x}]_2, \mathbf{A}_2, \mathbf{b}_2, \rho_2 \rangle = \langle [\mathbf{x}]_3, \mathbf{A}_3, \mathbf{b}_3, \rho_3 \rangle$$

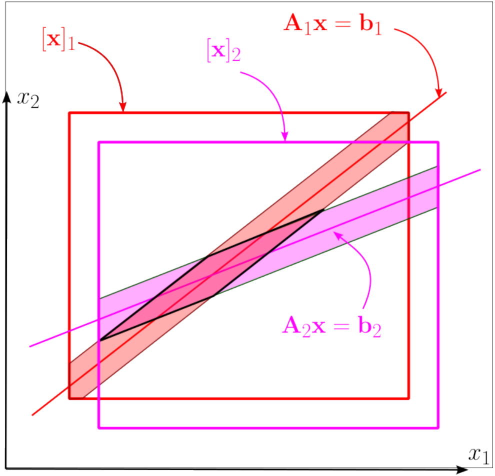
where

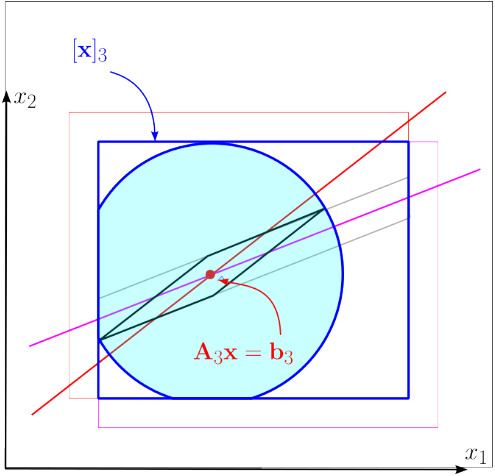
$$\begin{array}{rcl} [\mathbf{x}]_3 & = & [\mathbf{x}]_1 \cap [\mathbf{x}]_2 \\ \mathbf{A}_3 & = & \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \\ \mathbf{b}_3 & = & \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \\ \rho_3 & = & \frac{1}{\sin\theta} \sqrt{\rho_b^2 + \rho_a^2 + 2\rho_a \rho_b \cdot \cos\theta} \end{array}$$

where θ is the principle angle between $\mathrm{span}(\mathbf{A}_1)$ and $\mathrm{span}(\mathbf{A}_2)$.

Proposition

$$\mathbf{x} \in \langle \mathbf{x}_1 \rangle \text{ and } \mathbf{x} \in \langle \mathbf{x}_2 \rangle \Rightarrow \mathbf{x} \in \langle \mathbf{x}_1 \rangle \cap \langle \mathbf{x}_2 \rangle .$$


$[\mathbf{x}]_1$
$[\mathbf{x}]_2$
$\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$
$\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$
x_2
x_1


$[\mathbf{x}]_3$
x_2
x_1
$\mathbf{A}_3\mathbf{x} = \mathbf{b}_3$

Buche contractors

Consider again

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$, $m < n$.

A *buche contractor* $\mathscr{B}([\mathbf{x}])$ associated to $\mathbb{X}$ is an operator

$$\mathscr{B} : [\mathbf{x}] \rightarrow \langle \mathbf{x} \rangle = \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$

which satisfies

$$\mathbb{X} \cap [\mathbf{x}] \subset \langle \mathbf{x} \rangle .$$

Moreover, $\mathscr{B}$ is said to have an order k if for all nested sequence of boxes converging to $\mathbf{x} \in \mathbb{X}$,

$$\frac{h\left(\langle \mathbf{x} \rangle, \mathbb{X} \cap [\mathbf{x}]\right)}{\mathrm{rad}([\mathbf{x}])^k} \rightarrow 0.$$

Proposition. For $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$, the operator

$$\mathscr{B} : [\mathbf{x}] \to \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$

where

$$\begin{array}{rcl} \mathbf{b} & = & \mathbf{A}\bar{\mathbf{x}} - \mathbf{f}(\bar{\mathbf{x}}) \\ \mathbf{A} & = & \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) \\ \rho & = & \|\mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{A}^\mathsf{T}\right)^{-1}\| \cdot \mathsf{ub}\|[\mathbf{e}]\| \\ [\mathbf{e}] & = & \left(\mathbf{A} - \frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])\right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \end{array}$$

is a buche contractor of order 1.

Test-case

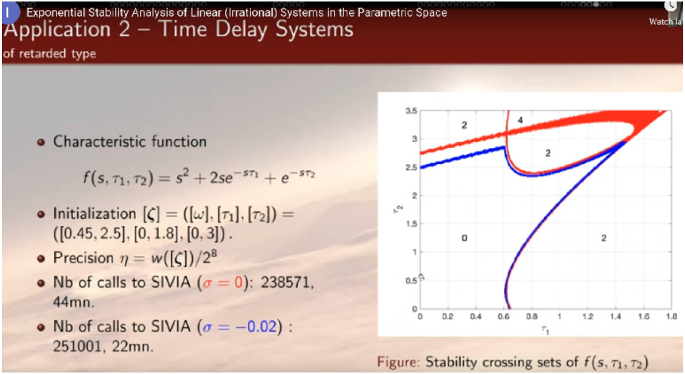
Consider the system :

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_3^2 + 2x_3 \sin(x_3 x_1) + \cos(x_3 x_2) \\ 2x_3 \cos(x_3 x_1) - \sin(x_3 x_2) \end{pmatrix} = \mathbf{0}$$

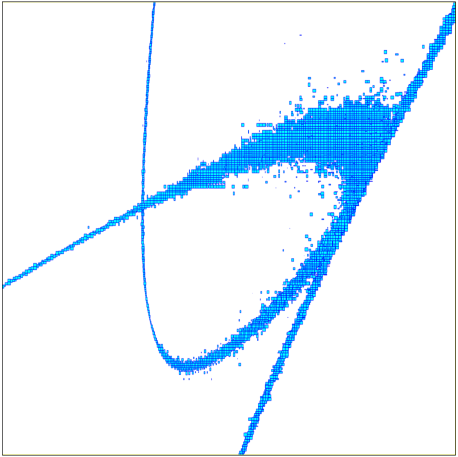
We want to characterize

$$\mathbb{P} = \{(x_1, x_2) | \exists x_3 \in [0, 10], \mathbf{f}(x_1, x_2, x_3) = \mathbf{0}\}.$$

With $[x_1] = [0, 2.5]$, $[x_2] = [1, 4]$, $[x_3] = [0, 10]$, a Matlab implementation, with a forward-backward contractor, and $\varepsilon = 2^{-8}$, yields:



https://youtu.be/DaR2NZZIV10?t=2453



$\varepsilon = 2^{-8}$, Codac [16] generated 43173 boxes.

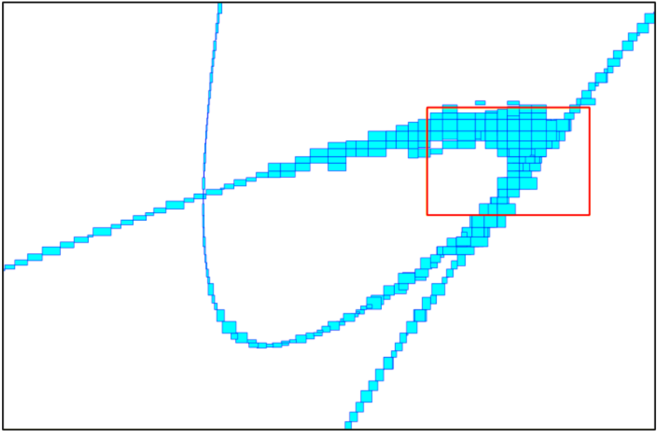
We still have a *Clustering effect*

Algorithm with buches

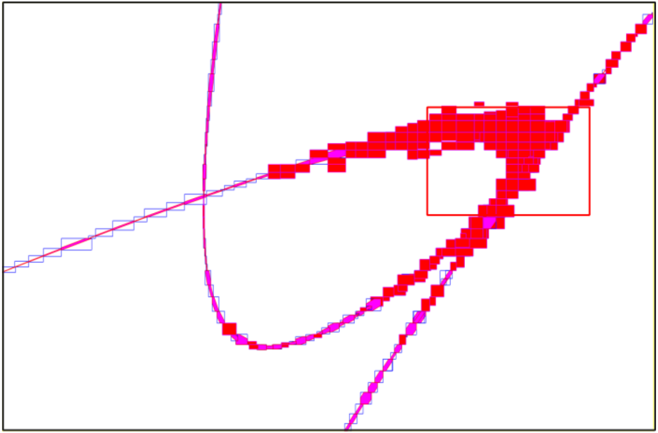
Initialization: $\mathscr{L} = \{[\mathbf{x}]\}$; $\mathbb{X}^+ = \{\}$.

Resolution.

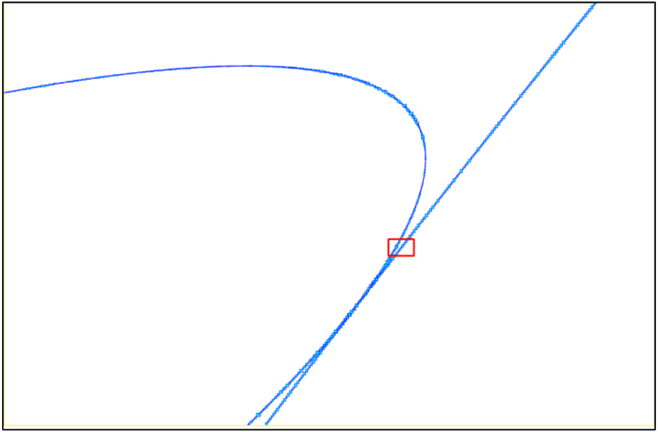
- **Contraction step**. Replace each $[\mathbf{x}] \in \mathscr{L}$, by the smallest box enclosing the buche $\langle\mathbf{x}\rangle = \mathscr{B}([\mathbf{x}])$.
- **Bisection step**. For each $[\mathbf{x}] \in \mathscr{L}$, if $\text{Vol}(\langle\mathbf{x}\rangle) < \varepsilon^n$ then push $\langle\mathbf{x}\rangle$ on $\mathbb{X}^+$, otherwise, bisect $[\mathbf{x}]$ and push the two resulting boxes in $\mathscr{L}$.



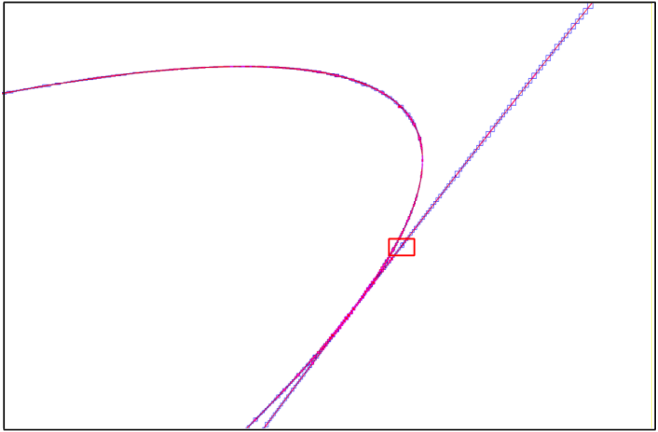
(a) Asymptotically minimal contractor



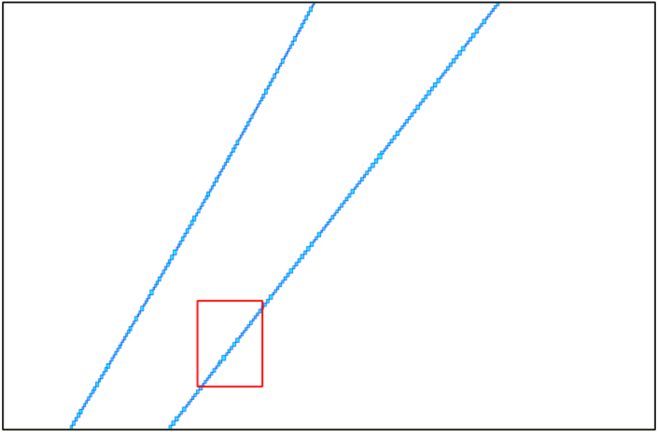
(a) Buche contractor



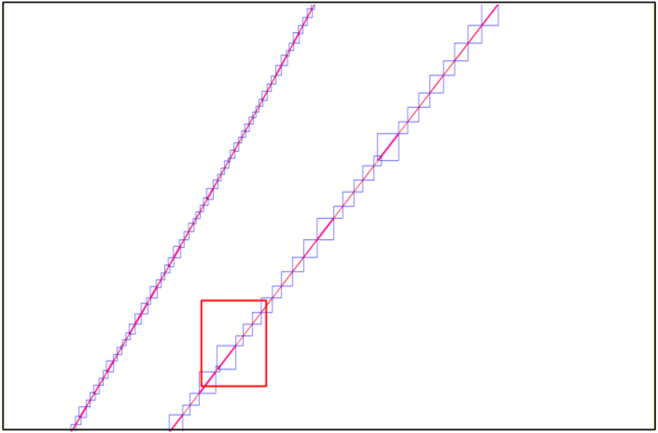
(b) Asymptotically minimal contractor



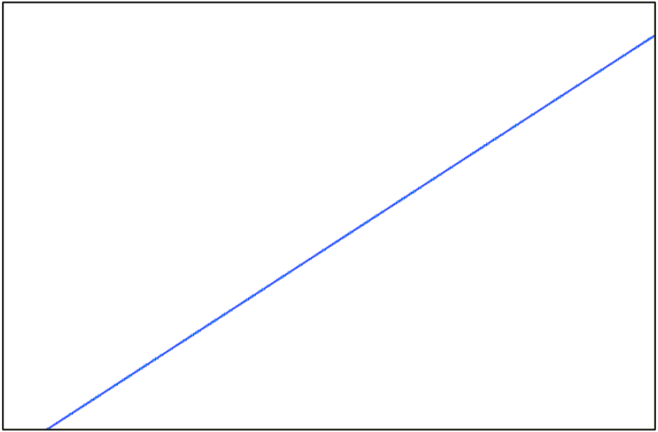
(b) Buche contractor



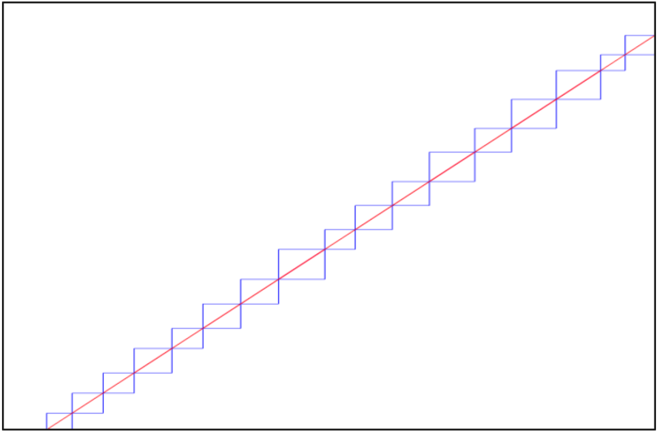
(c) Asymptotically minimal contractor



(c) Buche contractor



(d) Asymptotically minimal contractor



(d) Buche contractor

	Case (a)	Case (b)	Case (c)	Case (d)
$[\mathbf{x}]$	$\begin{pmatrix} [0,2] \\ [2,4] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.3,1.8] \\ [3,3.5] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.595,1.615] \\ [3.2,3.22] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.601,1.603] \\ [3.202,3.206] \\ [0,10] \end{pmatrix}$
ε	2^{-4}	2^{-8}	2^{-12}	2^{-16}
$T_1(s)$	0.51	0.84	0.11	0.19
$T_2(s)$	0.86	1.46	0.17	0.05
V_1	0.092	$2.05 \cdot 10^{-3}$	$3.06 \cdot 10^{-6}$	$4.18 \cdot 10^{-7}$
V_2	0.02	$1.93 \cdot 10^{-3}$	$7.2 \cdot 10^{-7}$	$5.78 \cdot 10^{-9}$
$\frac{V_1}{V_2}$	3.8	4.6	15.3	72.4

The code source of the test-case is based on the codac library [16] and can be found at:

https://www.ensta-bretagne.fr/jaulin/buche.html

References

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3. Contractor programming : [3][2]
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5. Asymptotically minimal contractor [7].
6. Order one approximation : [4][6]
7. Test-case : [19][10]

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