Calibrating magnetomers and accelerometers

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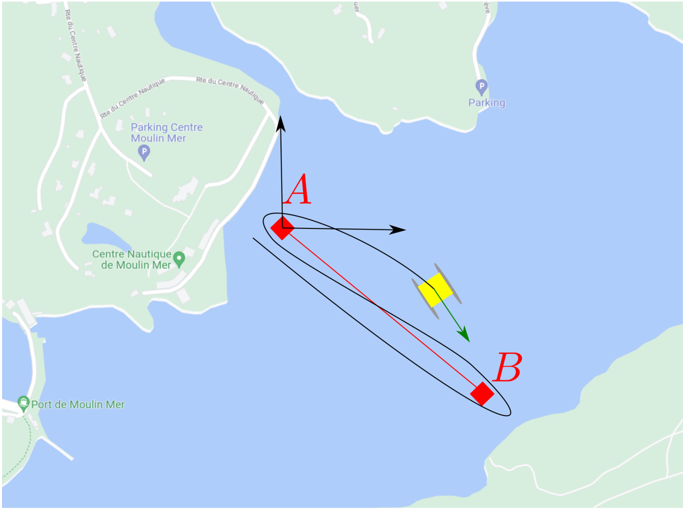
1. Calibration of the magnetometers


Rte du Centre Nautique
Rte du Centre Nautique
Rte du Centre Nautique
Parking Centre Moulin Mer
Parking
Centre Nautique de Moulin Mer
Port de Moulin Mer
A
B

In Brest, the inclination of the magnetic field is: $I = 64°$.
The declination is 1.4° and its intensity is : $\beta = 46.0\mu T$.
See: franceipgp_Jussieu.html

For a dipole,

$$I = \arctan(2\tan\lambda)$$

For simplicity, we assume that the declination is 0, *i.e.*, the magnetic field is pointing toward the North.

We assume that

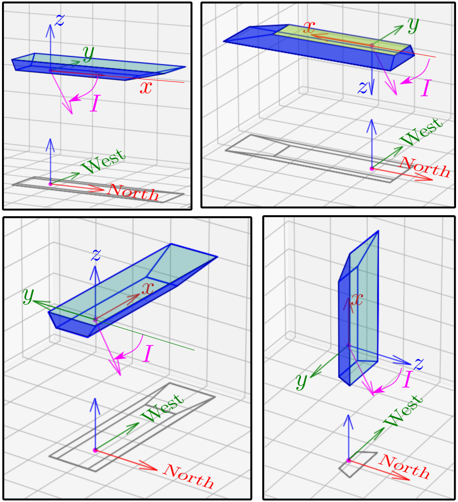
$$\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x} + \mathbf{b})$$

where $\mathbf{x} \in \mathbb{R}^3$ is the vector of magnetic measurements, and $\mathbf{y}$ is the corrected vector.

The calibration step searches for an estimation of $\mathbf{A}$ and $\mathbf{b}$.

We take four measurements $\mathbf{x}_N, \mathbf{x}_S, \mathbf{x}_W, \mathbf{x}_U$, with the robot

- Direction N : The robot is laid flat toward the North
- Direction S : The robot is laid toward the South, upside down
- Direction W : The robot is laid flat toward the West
- Direction U : The robot is set vertically with its z axis toward the North


z
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x
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x
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z
I
West
North

The four corresponding transformation matrices are

$$\mathbf{R}_N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{R}_W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The corrected magnetic field should be

$$\mathbf{y}_N = \begin{pmatrix} \beta \cdot \cos I \\ 0 \\ -\beta \cdot \sin I \end{pmatrix}, \qquad \mathbf{y}_S = \mathbf{R}_S^{\mathsf{T}} \cdot \mathbf{y}_N = \begin{pmatrix} -\beta \cdot \cos I \\ 0 \\ \beta \cdot \sin I \end{pmatrix}$$

$$\mathbf{y}_W = \mathbf{R}_W^{\mathsf{T}} \cdot \mathbf{y}_N = \begin{pmatrix} 0 \\ -\beta \cdot \cos I \\ -\beta \cdot \sin I \end{pmatrix}, \quad \mathbf{y}_U = \mathbf{R}_U^{\mathsf{T}} \cdot \mathbf{y}_N = \begin{pmatrix} -\beta \cdot \sin I \\ 0 \\ \beta \cdot \cos I \end{pmatrix}$$

where $\beta = 46\mu T$.

Since

$$\begin{aligned} \mathbf{0} &= \mathbf{y}_N + \mathbf{y}_S \\ &= \mathbf{A}^{-1}(\mathbf{x}_N + \mathbf{b}) + \mathbf{A}^{-1}(\mathbf{x}_S + \mathbf{b}) \end{aligned}$$

we get

$$(\mathbf{x}_N + \mathbf{b}) + (\mathbf{x}_S + \mathbf{b}) = \mathbf{0}$$

i.e.,

$$\mathbf{b} = -\frac{1}{2}(\mathbf{x}_N + \mathbf{x}_S).$$

Moreover

$$\begin{aligned}\mathbf{A}\mathbf{y}_N &= \mathbf{x}_N + \mathbf{b}\\ \mathbf{A}\mathbf{y}_W &= \mathbf{x}_W + \mathbf{b}\\ \mathbf{A}\mathbf{y}_U &= \mathbf{x}_U + \mathbf{b}\end{aligned}$$

yields

$$\mathbf{A}\cdot\underbrace{\begin{pmatrix}\mathbf{y}_N & \mathbf{y}_W & \mathbf{y}_U\end{pmatrix}}_{\mathbf{Y}} = \underbrace{\begin{pmatrix}\mathbf{x}_N+\mathbf{b} & \mathbf{x}_W+\mathbf{b} & \mathbf{x}_U+\mathbf{b}\end{pmatrix}}_{\mathbf{X}}$$

i.e.,

$$\mathbf{A} = \mathbf{X} \cdot \mathbf{Y}^{-1}$$

Algorithm MagCalib(In: $\mathbf{x}_N, \mathbf{x}_S, \mathbf{x}_W, \mathbf{x}_U$; Out :$\mathbf{A}, \mathbf{b}$)

$I = 64 \cdot \frac{\pi}{180}; \beta = 46.0 \cdot 10^{-6} T.$

$\mathbf{b} = -\frac{1}{2}\left(\mathbf{x}_N + \mathbf{x}_S\right)$

$$\mathbf{Y} = \beta \cdot \begin{pmatrix} \cos I & 0 & -\sin I \\ 0 & -\cos I & 0 \\ -\sin I & -\sin I & \cos I \end{pmatrix}$$

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_N + \mathbf{b} & \mathbf{x}_W + \mathbf{b} & \mathbf{x}_U + \mathbf{b} \end{pmatrix}$$

$\mathbf{A} = \mathbf{X} \cdot \mathbf{Y}^{-1}$

Return $\mathbf{A}$, $\mathbf{b}$

2. Calibration of the accelerometers

We assume that

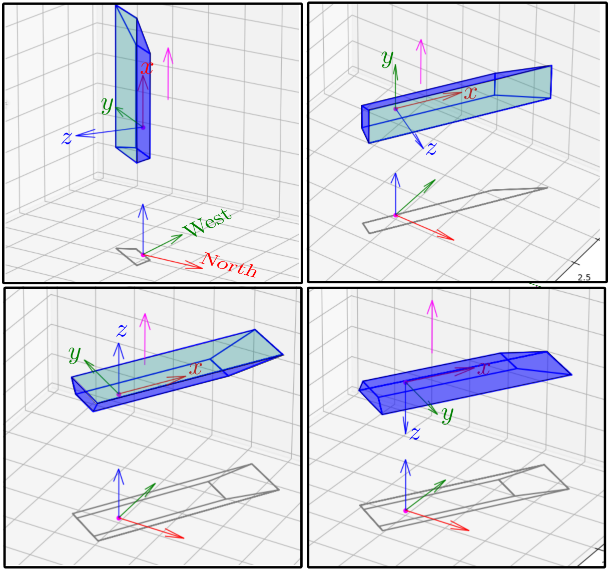
$$\mathbf{y} = \mathbf{A}^{-1}(\mathbf{x} + \mathbf{b})$$

where $\mathbf{x} \in \mathbb{R}^3$ is the acceleration vector which is measured, and $\mathbf{y}$ is gravity vector (oriented upward).
We assume the boat static so that the acceleration corresponds to the gravity.

The calibration step searches for an estimation of $\mathbf{A}$ and $\mathbf{b}$.

We take four measurements $\mathbf{x}_x, \mathbf{x}_y, \mathbf{x}_z, \mathbf{x}_{-z},$, with the robot. We do not care where the North is.

- Direction x : The robot is set vertically with its x axis looking upward
- Direction y : The robot is set with roll its y axis upward
- Direction z : The robot is laid flat with its z axis upward
- Direction $-z$: The robot is laid flat upside down with its z axis downward


x
y
z
West
North
y
x
z
2.5
z
y
x
x
y
z

The corresponding corrected acceleration vectors should be

$$\mathbf{y}_x = \begin{pmatrix} \beta \\ 0 \\ 0 \end{pmatrix}, \mathbf{y}_y = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}, \mathbf{y}_z = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \mathbf{y}_{-z} = \begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix}$$

where $\beta = 9.8 m \cdot s^{-2}$.

Since

$$\begin{aligned} \mathbf{0} &= \mathbf{y}_z + \mathbf{y}_{-z} \\ &= \mathbf{A}^{-1}(\mathbf{x}_z + \mathbf{b}) + \mathbf{A}^{-1}(\mathbf{x}_{-z} + \mathbf{b}) \end{aligned}$$

we get

$$(\mathbf{x}_z + \mathbf{b}) + (\mathbf{x}_z + \mathbf{b}) = \mathbf{0}$$

i.e.,

$$\mathbf{b} = -\frac{1}{2}(\mathbf{x}_z + \mathbf{x}_{-z}).$$

Moreover

$$\begin{aligned} \mathbf{A}\mathbf{y}_x &= \mathbf{x}_z + \mathbf{b} \\ \mathbf{A}\mathbf{y}_y &= \mathbf{x}_y + \mathbf{b} \\ \mathbf{A}\mathbf{y}_z &= \mathbf{x}_z + \mathbf{b} \end{aligned}$$

yields

$$\mathbf{A} \cdot \underbrace{\begin{pmatrix} \mathbf{y}_x & \mathbf{y}_y & \mathbf{y}_z \end{pmatrix}}_{\beta \mathbf{I}} = \underbrace{\begin{pmatrix} \mathbf{x}_x + \mathbf{b} & \mathbf{x}_y + \mathbf{b} & \mathbf{x}_z + \mathbf{b} \end{pmatrix}}_{\mathbf{X}}$$

Thus

$$\mathbf{A} = \frac{1}{\beta}\mathbf{X}$$

	Algorithm MagAccelero(In: $\mathbf{x}_x, \mathbf{x}_y, \mathbf{x}_z, \mathbf{x}_{-z}$; Out :$\mathbf{A}, \mathbf{b}$)
1	$\beta = 9.8 m \cdot s^{-2}$
2	$\mathbf{b} = -\frac{1}{2}(\mathbf{x}_z + \mathbf{x}_{-z})$
3	$\mathbf{X} = \begin{pmatrix} \mathbf{x}_x + \mathbf{b} & \mathbf{x}_y + \mathbf{b} & \mathbf{x}_z + \mathbf{b} \end{pmatrix}$
4	$\mathbf{A} = \frac{1}{\beta}\mathbf{X}$
5	Return $\mathbf{A}$, $\mathbf{b}$

3. IMU

The boat is moving in the waves.
An IMU estimates the Euler angles from a magnetometer and an accelerometer.

These Euler angles φ, θ, ψ have to be estimated from

1. the magnetic field $\mathbf{y}_1$ and
2. its acceleration $\mathbf{a}_1$, both given in the boat frame.

Estimate the vertical

Assumption. In average the accelerometers returns the gravity vector $\mathbf{g}_1$ is the robot frame, i.e.,

$$\mathbf{a}_1 = \mathbf{g}_1 + \beta$$

where β is a centered noise.
The assumption is false if the boat turns around a buoy for long.

The sensors return $\mathbf{a}_1$ and ω_1 in the robot frame.
The gravity vector $\mathbf{g}_1$ returns the vertical.

We have

$$\mathbf{g}_0 = \mathbf{R}\mathbf{g}_1$$

Differentiate with respect to t:

$$\mathbf{0} = \dot{\mathbf{R}}\mathbf{g}_1 + \mathbf{R}\dot{\mathbf{g}}_1$$

Thus

$$\begin{aligned} \dot{\mathbf{g}}_1 &= -\mathbf{R}^\mathsf{T}\dot{\mathbf{R}}\mathbf{g}_1 \\ &= -\omega_1 \wedge \mathbf{g}_1 \end{aligned}$$

We have

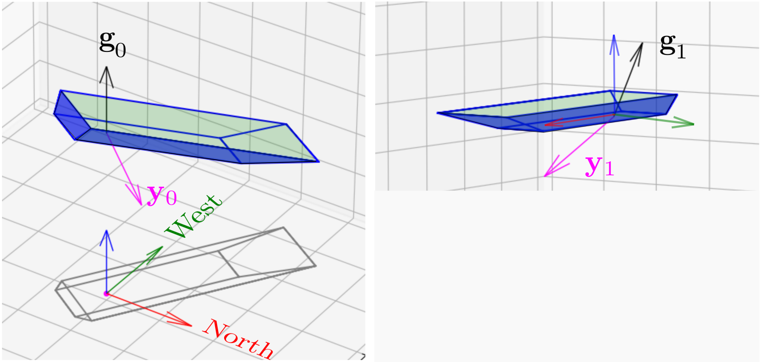
$$\begin{array}{rcll} \mathbf{g}_1(k+1) & = & \underbrace{(\mathbf{I} - dt\boldsymbol{\omega}_1(k)\wedge)}_{=\mathbf{A}(k)} \cdot \mathbf{g}_1(k) & \text{(prediction)} \\ \mathbf{a}_1(k) & = & \mathbf{g}_1(k) + \beta(k) & \text{(observation)} \end{array}$$

with a large variance for $\beta(k)$, which means that we measure $\mathbf{g}_1$ (through the acceleration $\mathbf{a}_1$), only in average.

A Luenberger observer could be

$$\hat{\mathbf{g}}_1(k+1) = \lambda \cdot (\mathbf{I} - dt\omega_1(k)\wedge) \cdot \hat{\mathbf{g}}_1(k) + (1-\lambda) \cdot \mathbf{a}_1(k+1)$$

with $\lambda = 0.99$.



We now assume that we are able to have a good estimation of $\mathbf{g}_1$

For simplicity, we assume that $\mathbf{y}$ and $\mathbf{g}$ have been normalized, *i.e.*, their norm are both equal to 1. Moreover, we assume that sensors are well calibrated.
In $\mathscr{R}_0$, we have

$$\mathbf{y}_0 = \begin{pmatrix} \cos I \\ 0 \\ -\sin I \end{pmatrix}, \ \mathbf{g}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Step 1. From $\mathbf{g}_1$, we get an estimation of the bank φ (roll angle) and the elevation θ (pitch angle).

We have

$$\hat{\varphi} = \arcsin\left((0,1,0)\cdot \mathbf{g}_1\right)$$

and

$$\hat{\theta} = -\arcsin\left((1,0,0)\cdot \mathbf{g}_1\right)$$

Step 2. For the heading, we first compute the rotation which leads our boat to the nearest horizontal orientation.

If $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors, the rotation $\mathbf{R}$ which minimizes $\|\mathbf{R}-\mathbf{I}\|$ and transforms $\mathbf{u}$ to $\mathbf{v}$ is given by

$$\mathbf{R}_{uv}(\mathbf{u},\mathbf{v}) = \mathrm{Exp}\left(\arccos(\mathbf{u}^\mathsf{T}\cdot\mathbf{v})\cdot\frac{\mathbf{u}\wedge\mathbf{v}}{\|\mathbf{u}\wedge\mathbf{v}\|}\right)$$

The rotation which transforms $\mathbf{g}_1$ to $\mathbf{g}_0$ is given by

$$\mathbf{R}_h = \mathbf{R}_{uv}(\mathbf{g}_1,\mathbf{g}_0).$$

Step 3. Provide an estimation $\hat{\psi}$ of the heading angle ψ.

We form the vector

$$\mathbf{y}_h = \mathbf{R}_h \mathbf{y}_1$$

which corresponds to the magnetic field measured by the horizontal avatar.
The heading is then obtained by

$$\hat{\psi} = -\mathrm{atan2}\left(y_{h2}, y_{h1}\right).$$

Algorithm EulerAngles(In: $\mathbf{a}_1, \mathbf{y}_1$; out: $\hat{\varphi}, \hat{\theta}, \hat{\psi}$)

$$\mathbf{g}_0 = (0,0,1)^\mathsf{T}, \mathbf{g}_1^n = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|}, \mathbf{y}_1^n = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|}$$

$$\hat{\varphi} = \arcsin\left((0,1,0)\cdot \mathbf{g}_1^n\right)$$

$$\hat{\theta} = -\arcsin\left((1,0,0)\cdot \mathbf{g}_1^n\right)$$

$$\mathbf{R}_h = \mathbf{R}_{uv}(\mathbf{g}_1^n, \mathbf{g}_0)$$

$$\mathbf{y}_h = \mathbf{R}_h \mathbf{y}_1^n$$

$$\hat{\psi} = -\mathrm{atan2}(y_{h2}, y_{h1})$$

With magnetometers, accelerometers and without gyro

Algorithm EulerAngles(In: $\mathbf{a}_1, \mathbf{y}_1, \omega_1, \hat{\mathbf{g}}_1$; out: $\hat{\varphi}, \hat{\theta}, \hat{\psi}, \hat{\mathbf{g}}_1$)

$\mathbf{g}_0 = (0,0,1)^\mathsf{T}$, $\mathbf{y}_1^n = \frac{\mathbf{y}_1}{\|\mathbf{y}_1\|}$

$\hat{\mathbf{g}}_1 = \lambda \cdot (\mathbf{I} - dt\omega_1 \wedge) \cdot \hat{\mathbf{g}}_1 + (1-\lambda) \cdot \mathbf{a}_1$, with $\lambda \simeq 1$

$\mathbf{g}_1^n = \frac{\hat{\mathbf{g}}_1}{\|\hat{\mathbf{g}}_1\|}$

$\hat{\varphi} = \arcsin\left((0,1,0) \cdot \mathbf{g}_1^n\right)$

$\hat{\theta} = -\arcsin\left((1,0,0) \cdot \mathbf{g}_1^n\right)$

$\mathbf{R}_h = \mathbf{R}_{uv}(\mathbf{g}_1^n, \mathbf{g}_0)$

$\mathbf{y}_h = \mathbf{R}_h \mathbf{y}_1^n$

$\hat{\psi} = -\mathrm{atan2}(y_{h2}, y_{h1})$

With magnetometers, accelerometers and gyro

4. Magnetometer autocalibration

Magnetometer autocalibration is the process of determining hard- and soft-iron calibration parameters from arbitrary 3D orientation data, without requiring known attitude or external references.

Equivalently, we want to find a transformation that converts the distorted ellipsoid into a sphere centered at the origin, to correct for hard- and soft-iron effects, without taking care of the rotation part which can be known in advance.

We have many points $(x_1(i), x_2(i), x_3(i))$, $i \geq 1$, collected by the magnetometer.

All belong to an ellipse:

$$\underbrace{p_1x_1^2 + p_2x_1^2 + p_3x_1^2 + p_4x_1x_2 + p_5x_1x_3 + p_6x_2x_3 + p_7x_1 + p_8x_2 + p_9x_3}_{\mathbf{f_p}(\mathbf{x})} = 1.$$

i.e.,

$$\left(x_1^2|x_2^2|x_3^2|x_1x_2|x_1x_2|x_1x_3|x_2x_3|x_1|x_2|x_3\right) \cdot \mathbf{p} = 1$$

We should have

$$\underbrace{\begin{pmatrix} x_1^2(1) & x_2^2(1) & x_3^2(1) & x_1(1)x_2(1) & x_1(1)x_3(1) & x_2(1)x_3(1) & x_1(1) & x_2(\\ x_1^2(2) & x_2^2(2) & x_3^2(2) & x_1(2)x_2(2) & x_1(2)x_3(2) & x_2(2)x_3(2) & x_1(2) & x_2(\\ x_1^2(3) & x_2^2(3) & x_3^2(3) & x_1(3)x_2(3) & x_1(3)x_3(3) & x_2(3)x_3(3) & x_1(3) & x_2(\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}}_{\mathbf{M}}$$

A least-square solution is

$$\mathbf{p} = \left(\mathbf{M}^{\mathsf{T}}\mathbf{M}\right)^{-1} \cdot \mathbf{M}^{\mathsf{T}} \cdot \mathbf{1}$$

The ellipsoid can be written in matrix form

$$\underbrace{(\mathbf{x}-\mathbf{b})^{\mathsf{T}}\mathbf{Q}(\mathbf{x}-\mathbf{b})}_{\mathbf{f_p}(\mathbf{x})} = 1$$

The gradient for both expressions for $\mathbf{f_p}(\mathbf{x})$ should be equal.

$$\begin{aligned}\frac{\partial \mathbf{f_p}(\mathbf{x})}{\partial \mathbf{x}} &= p_1x_1^2 + p_2x_2^2 + p_3x_3^2 + p_4x_1x_2 + p_5x_1x_3 + p_6x_2x_3 + p_7x_1 + p_8x \\ &= (2p_1x_1 + p_4x_2 + p_5x_3 + p_7 | p_4x_1 + 2p_2x_2 + p_6x_3 + p_8 | p_5x_1 + p_6x_2 \\ &= (x_1|x_2|x_3)\begin{pmatrix} 2p_1 & p_4 & p_5 \\ p_4 & 2p_2 & p_6 \\ p_5 & p_6 & 2p_3 \end{pmatrix} + (p_7|p_8|p_9)\end{aligned}$$

And also

$$
\begin{aligned}
\frac{\partial \mathbf{f_p}(\mathbf{x})}{\partial \mathbf{x}} &= \frac{\partial (\mathbf{x}-\mathbf{b})^\mathsf{T}\mathbf{Q}(\mathbf{x}-\mathbf{b})}{\partial \mathbf{x}} \\
&= 2(\mathbf{x}-\mathbf{b})^\mathsf{T}\mathbf{Q} \\
&= 2\mathbf{x}^\mathsf{T}\mathbf{Q} - 2\mathbf{b}^\mathsf{T}\mathbf{Q}
\end{aligned}
$$

By identification, we get

$$\mathbf{Q} = \frac{1}{2}\begin{pmatrix} 2p_1 & p_4 & p_5 \\ p_4 & 2p_2 & p_6 \\ p_5 & p_6 & 2p_3 \end{pmatrix}$$

and

$$\mathbf{b} = -\frac{1}{2}\mathbf{Q}^{-1} \cdot \begin{pmatrix} p_7 \\ p_8 \\ p_9 \end{pmatrix}$$

To transform the ellipse to a sphere we take the transformation

$$\mathbf{y} = \sqrt{\mathbf{Q}} \cdot (\mathbf{x} - \mathbf{b})$$