

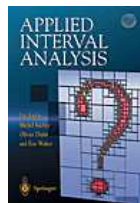
Interval analysis for proving properties
of dynamical systems;
application to sailboat robotics

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<http://www.ensta-bretagne.fr/jaulin/>

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Mathematics Colloquium



1 Interval analysis

Problem. Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$\begin{aligned} [-1, 3] + [2, 5] &= [1, 8], \\ [-1, 3] \cdot [2, 5] &= [-5, 15], \\ \text{abs}([-7, 1]) &= [0, 7] \end{aligned}$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 \\ + \sin x_1 \cdot \sin x_2 + 2$$

is

$$[f]([x_1], [x_2]) = [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos [x_2] \\ + \sin [x_1] \cdot \sin [x_2] + 2.$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^+ \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

2 Computing with sets

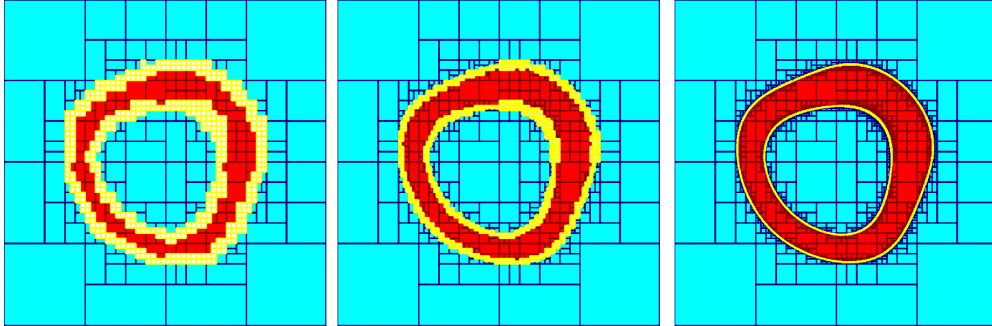
Subsets $\mathbb{X} \subset \mathbb{R}^n$ can be bracketted by subpavings :

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

which can be obtained using interval calculus

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 + \sin(x_1 + x_2) \in [4, 9]\}.$$



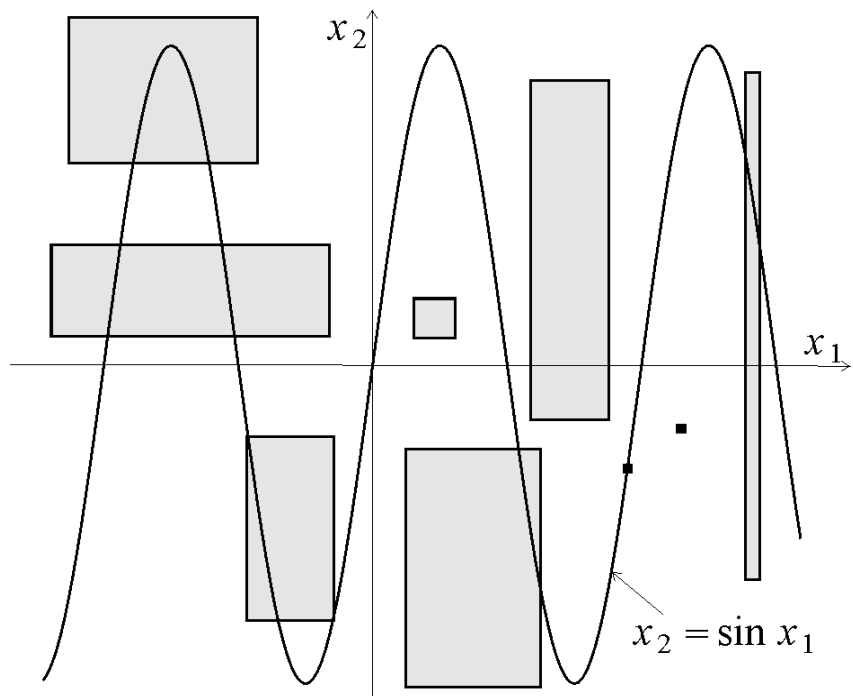
3 Contractors

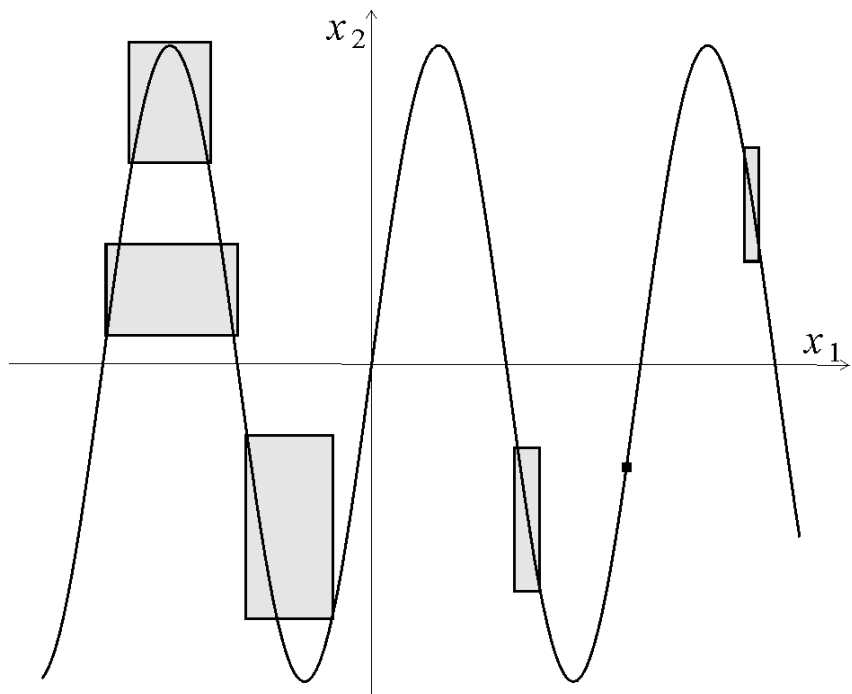
The operator $\mathcal{C} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

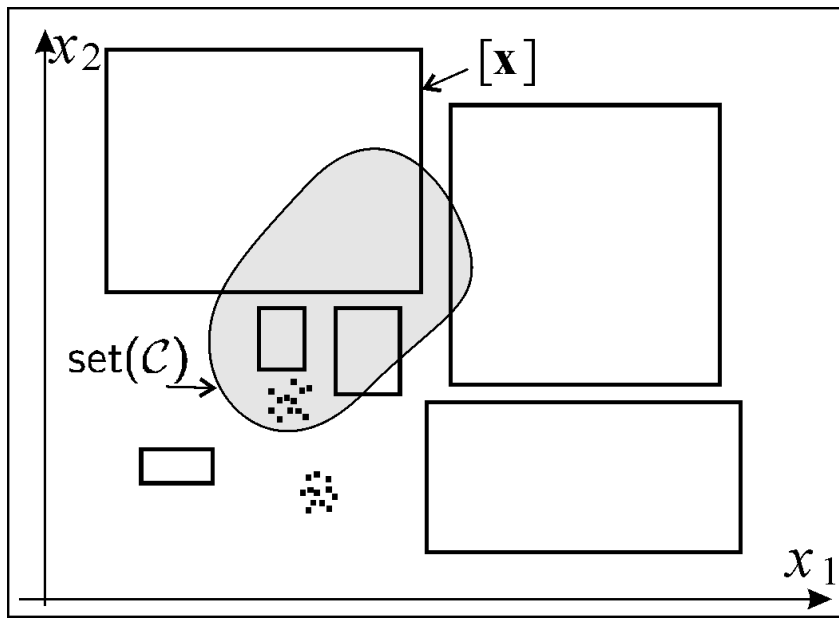
$$\begin{cases} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ \mathbf{x} \in [\mathbf{x}] \text{ and } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathcal{C}([\mathbf{x}]) & \text{(consistence)} \end{cases}$$

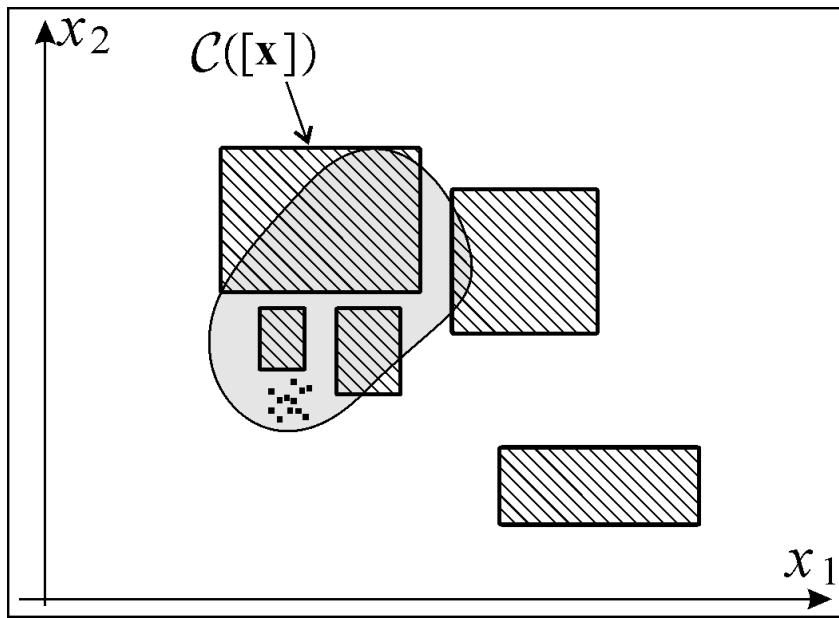
Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$









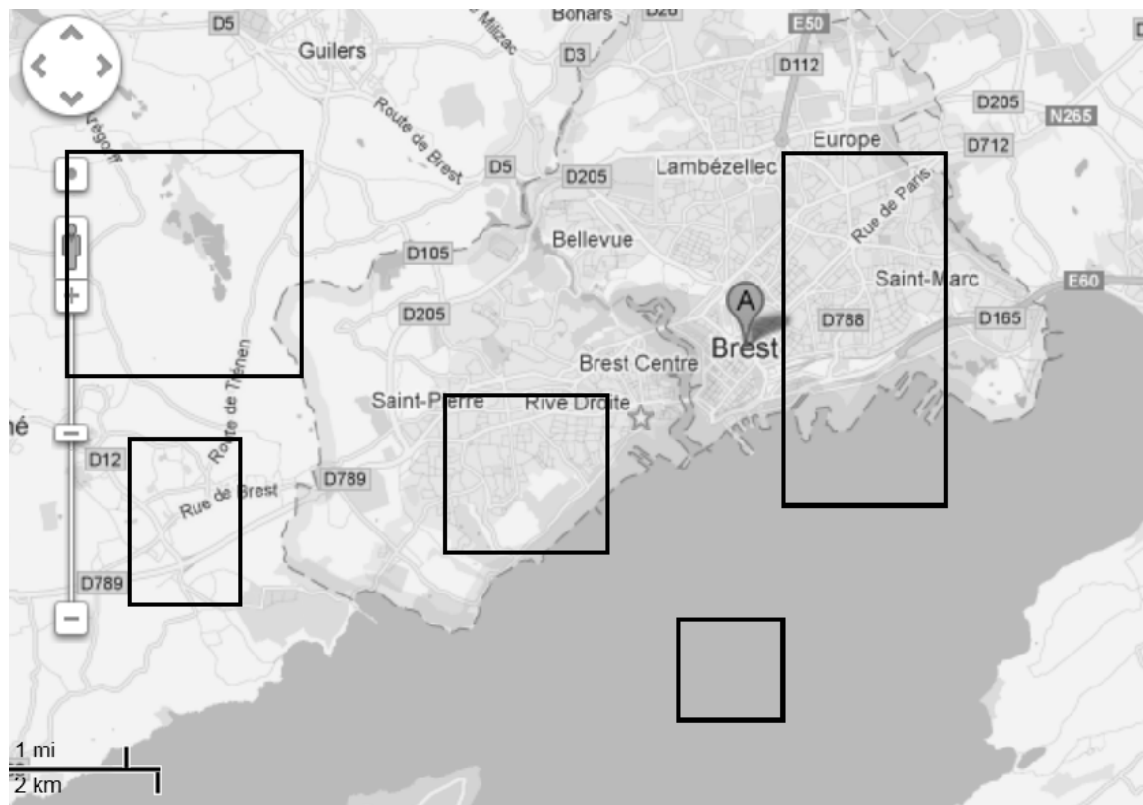
$\mathcal{C}$ is <i>monotonic</i> if	$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}])$
$\mathcal{C}$ is <i>idempotent</i> if	$\mathcal{C}(\mathcal{C}([\mathbf{x}])) = \mathcal{C}([\mathbf{x}])$

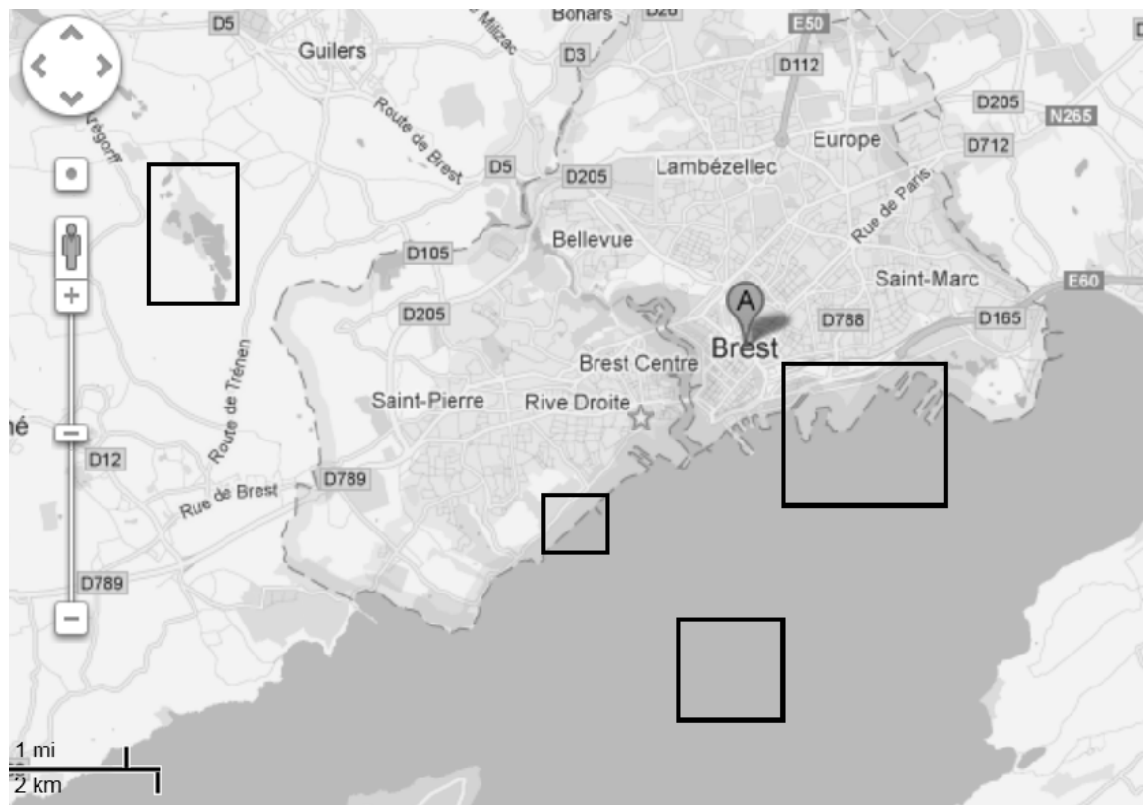
Contractor algebra

intersection	$(\mathcal{C}_1 \cap \mathcal{C}_2) ([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1 ([\mathbf{x}]) \cap \mathcal{C}_2 ([\mathbf{x}])$
union	$(\mathcal{C}_1 \cup \mathcal{C}_2) ([\mathbf{x}]) \stackrel{\text{def}}{=} [\mathcal{C}_1 ([\mathbf{x}]) \cup \mathcal{C}_2 ([\mathbf{x}])]$
composition	$(\mathcal{C}_1 \circ \mathcal{C}_2) ([\mathbf{x}]) \stackrel{\text{def}}{=} \mathcal{C}_1 (\mathcal{C}_2 ([\mathbf{x}]))$
reiteration	$\mathcal{C}^\infty \stackrel{\text{def}}{=} \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \dots$

Contractor associated with a database

The robot with coordinates (x_1, x_2) is in the water.





Building contractors for equations

Consider the primitive equation

$$x_1 + x_2 = x_3$$

with $x_1 \in [x_1]$, $x_2 \in [x_2]$, $x_3 \in [x_3]$.

We have

$$x_3 = x_1 + x_2 \Rightarrow x_3 \in [x_3] \cap ([x_1] + [x_2]) \quad // \text{ forward}$$

$$x_1 = x_3 - x_2 \Rightarrow x_1 \in [x_1] \cap ([x_3] - [x_2]) \quad // \text{ backward}$$

$$x_2 = x_3 - x_1 \Rightarrow x_2 \in [x_2] \cap ([x_3] - [x_1]) \quad // \text{ backward}$$

The contractor associated with $x_1 + x_2 = x_3$ is thus

$$\mathcal{C} \left(\begin{array}{c} [x_1] \\ [x_2] \\ [x_3] \end{array} \right) = \left(\begin{array}{c} [x_1] \cap ([x_3] - [x_2]) \\ [x_2] \cap ([x_3] - [x_1]) \\ [x_3] \cap ([x_1] + [x_2]) \end{array} \right)$$

4 Solver

Example. Solve the system

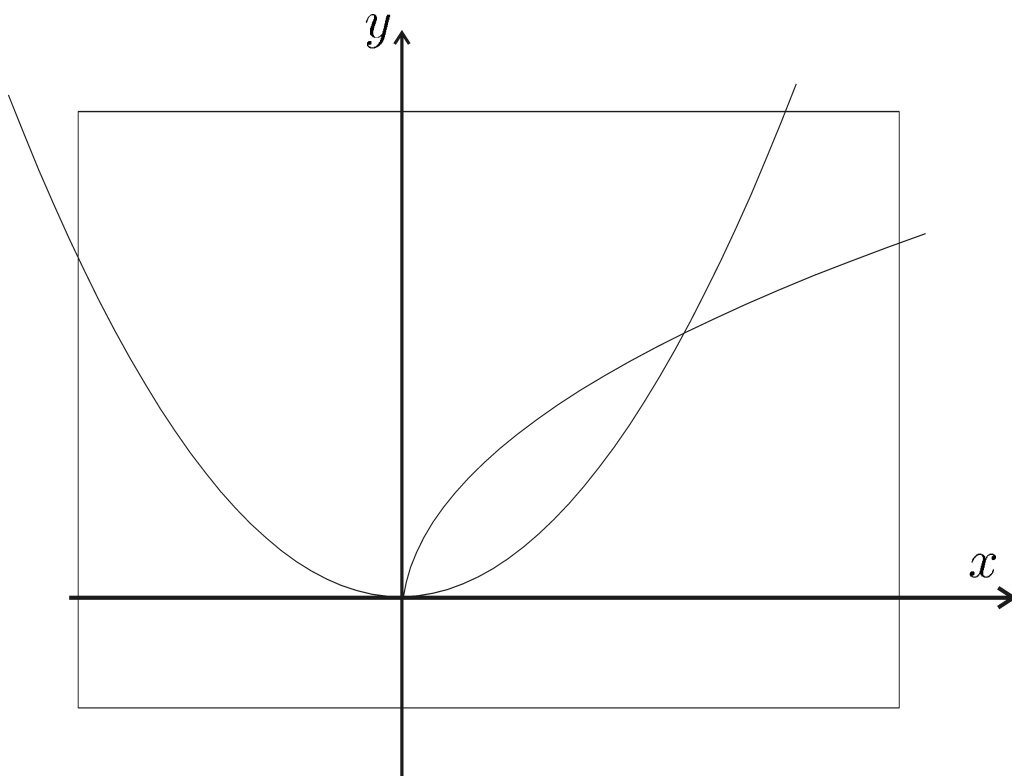
$$y = x^2$$

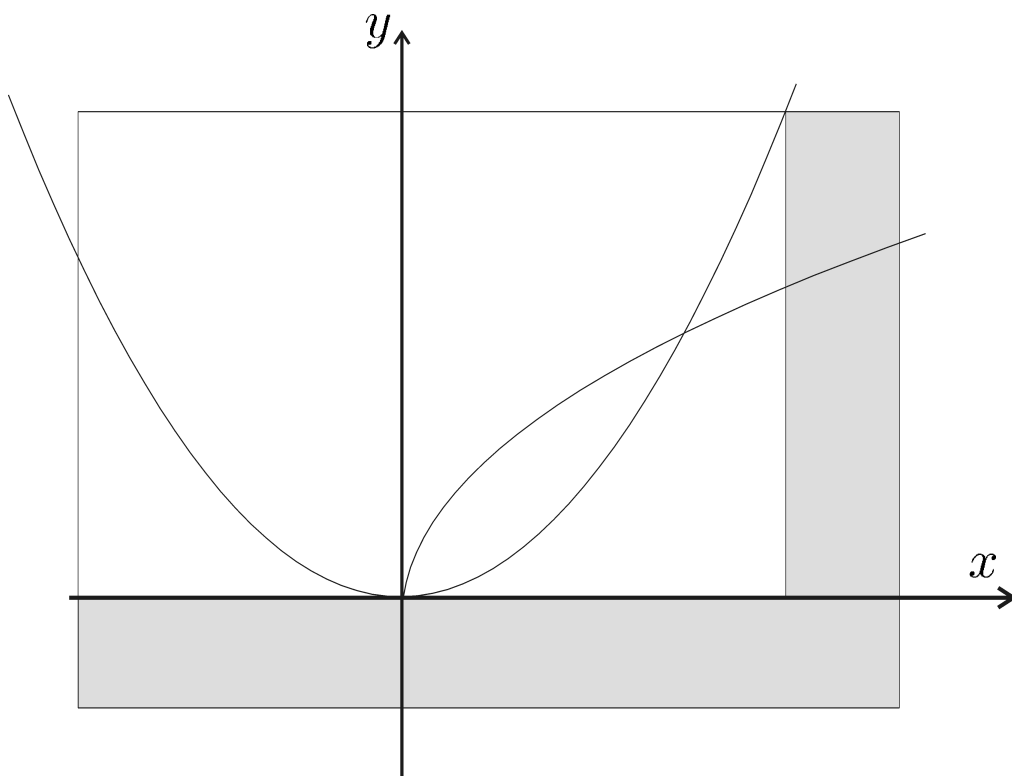
$$y = \sqrt{x}.$$

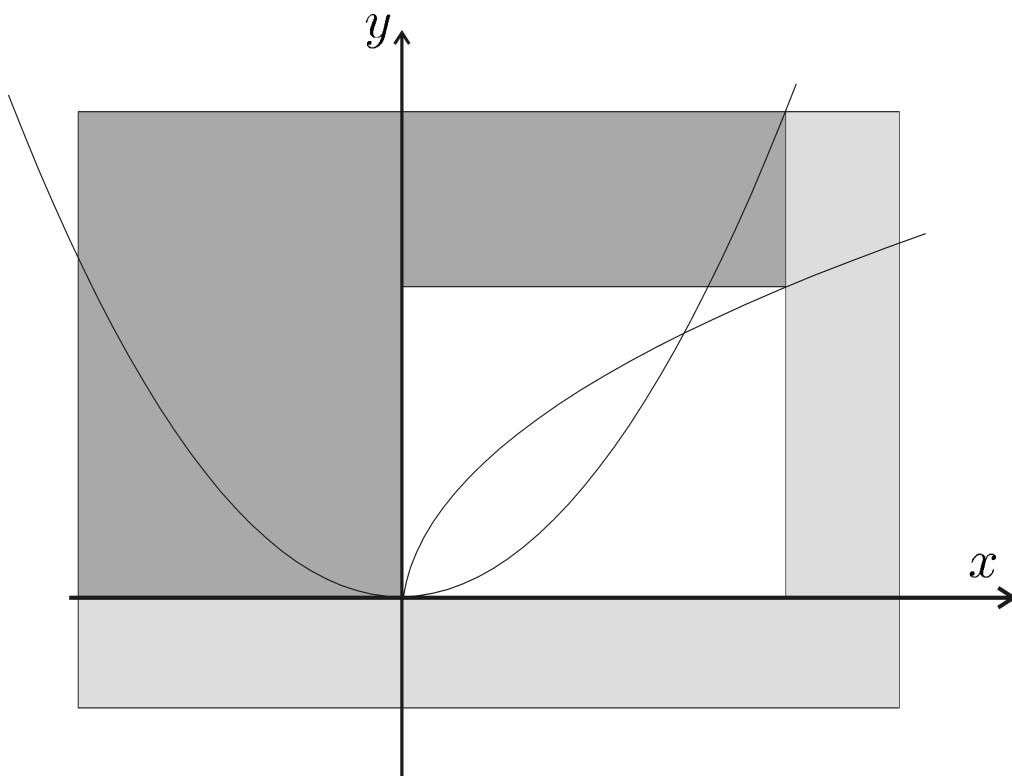
We build two contractors

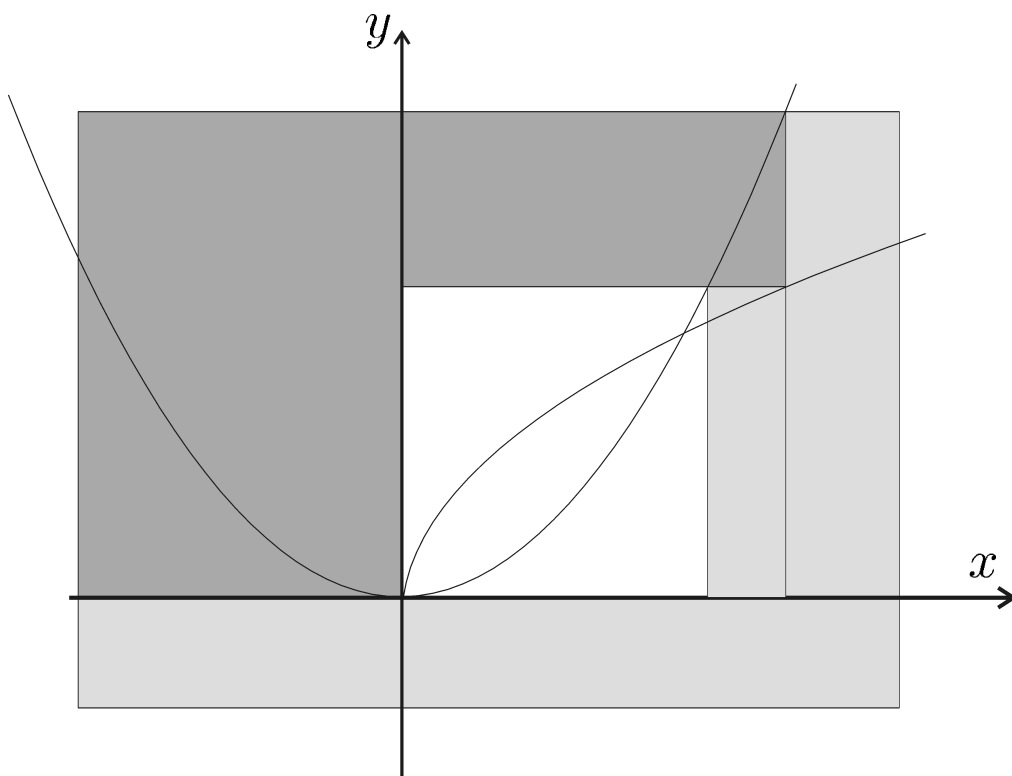
$$\mathcal{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated with } y = x^2$$

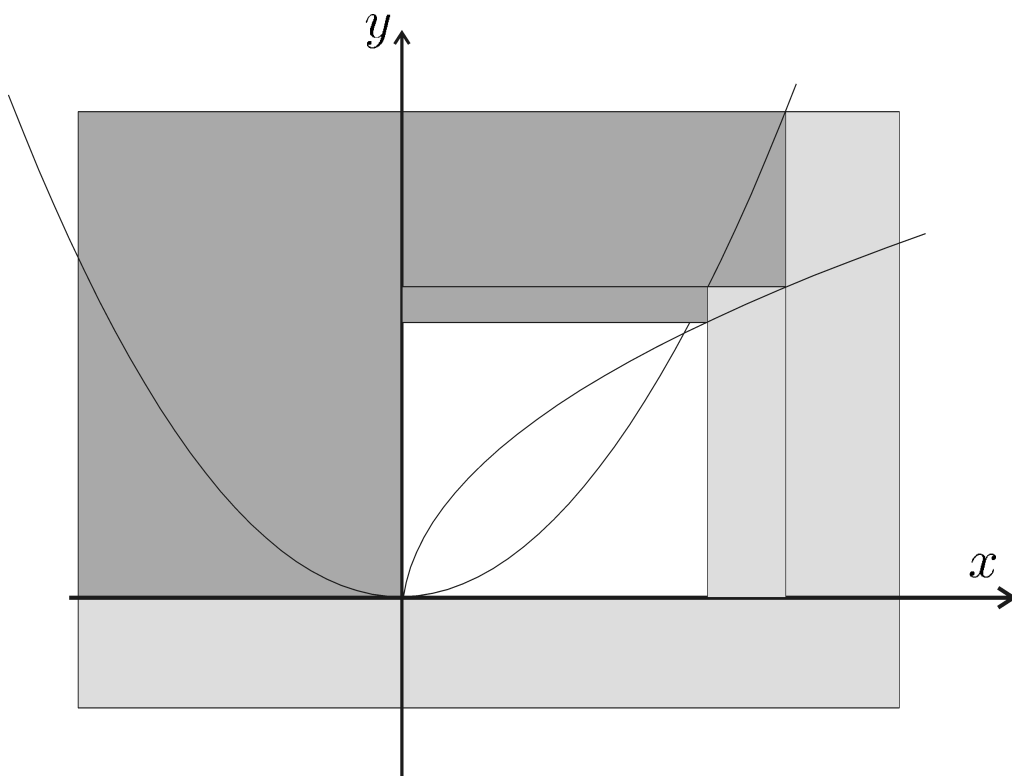
$$\mathcal{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated with } y = \sqrt{x}$$

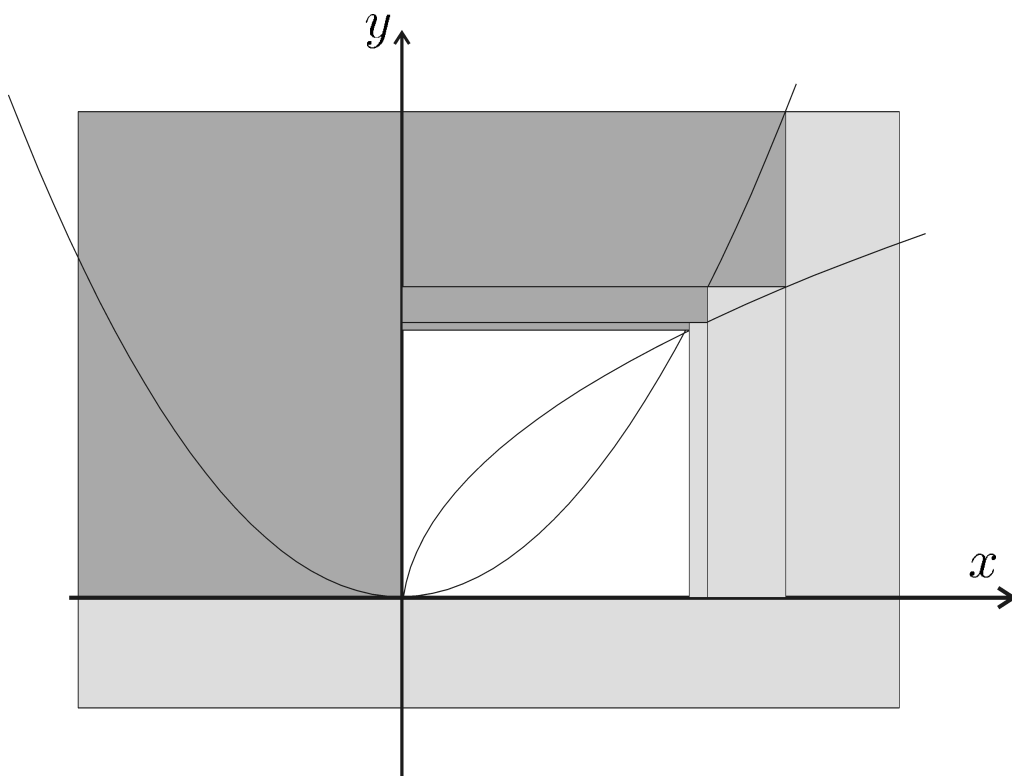


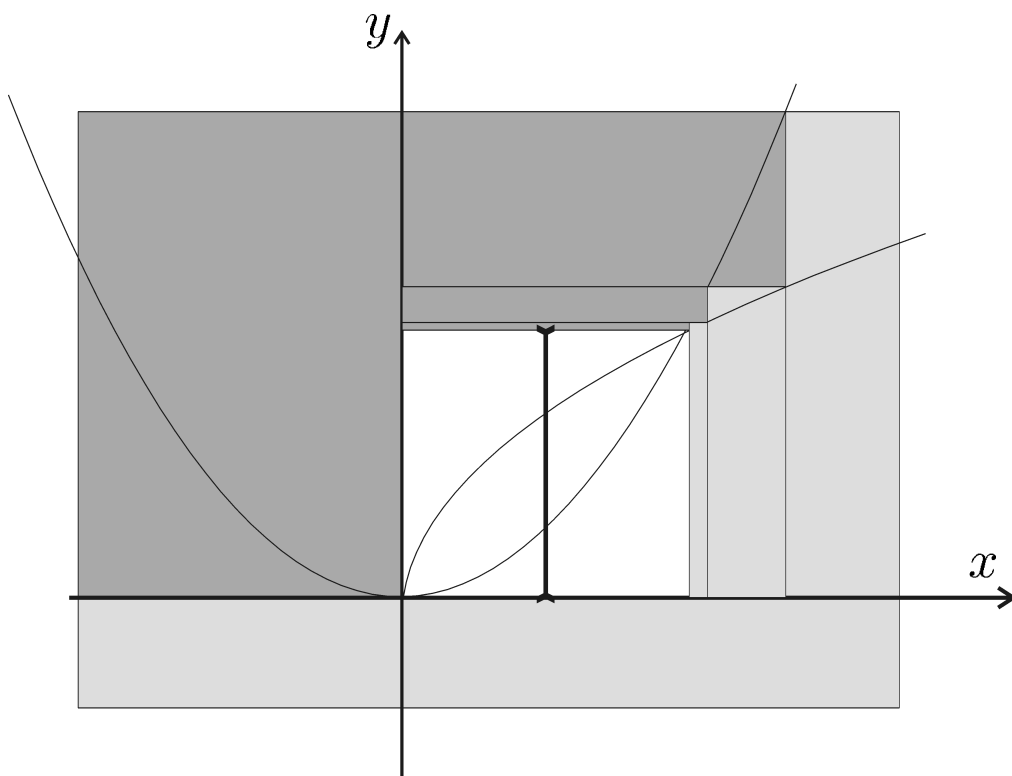


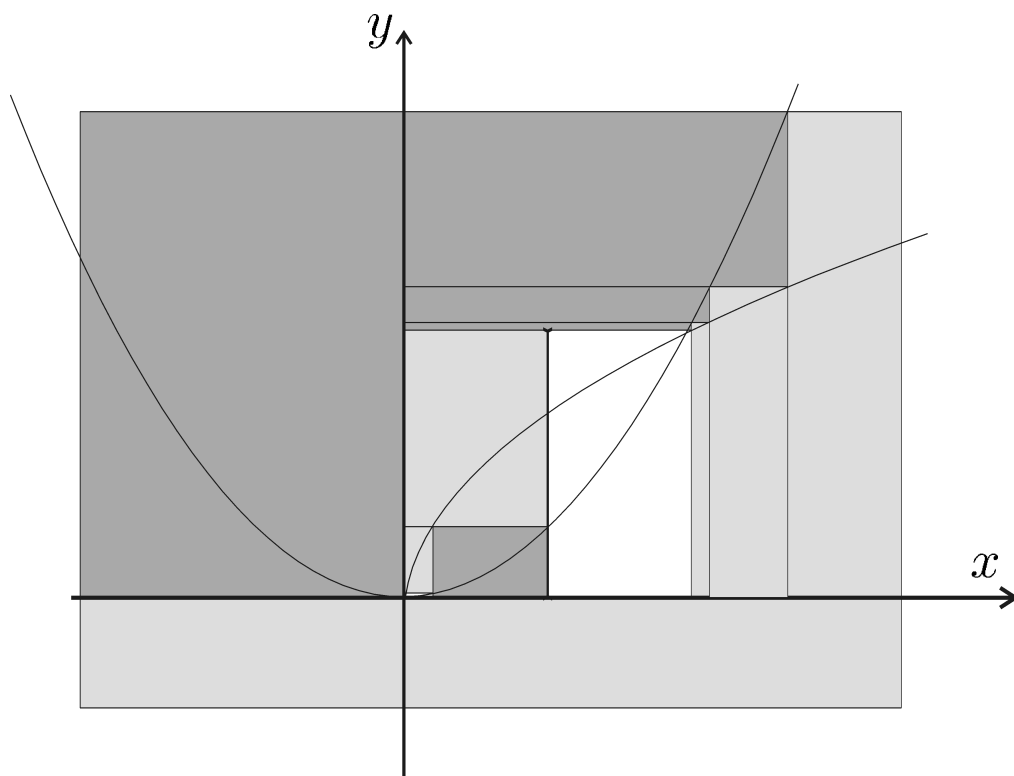


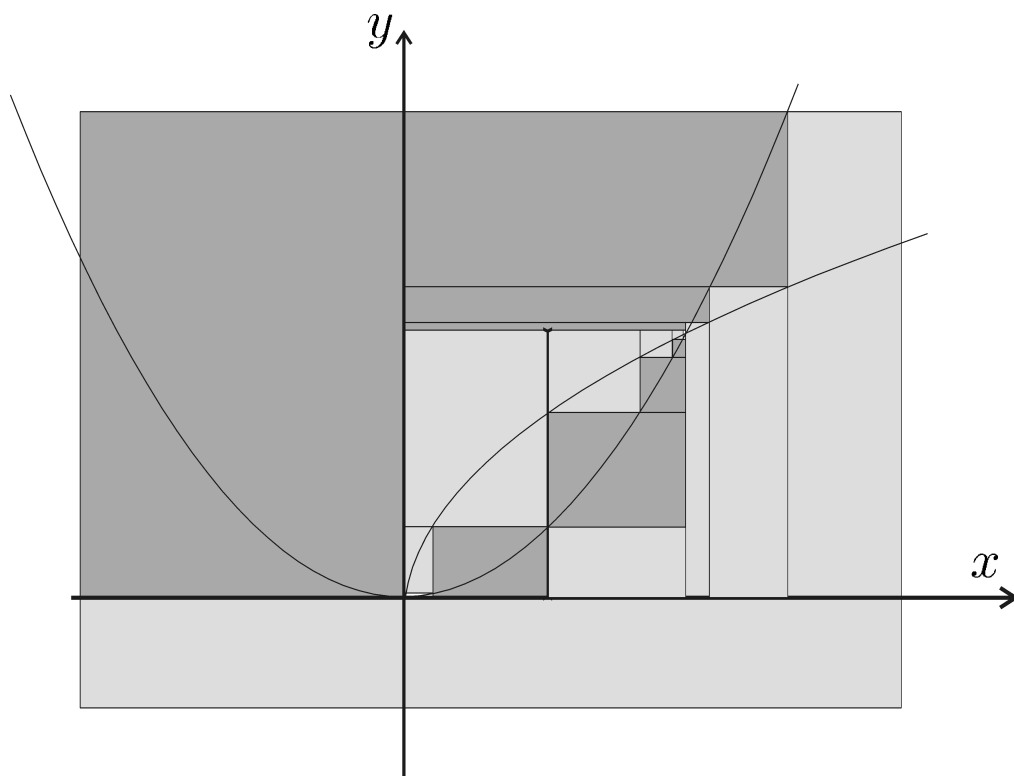






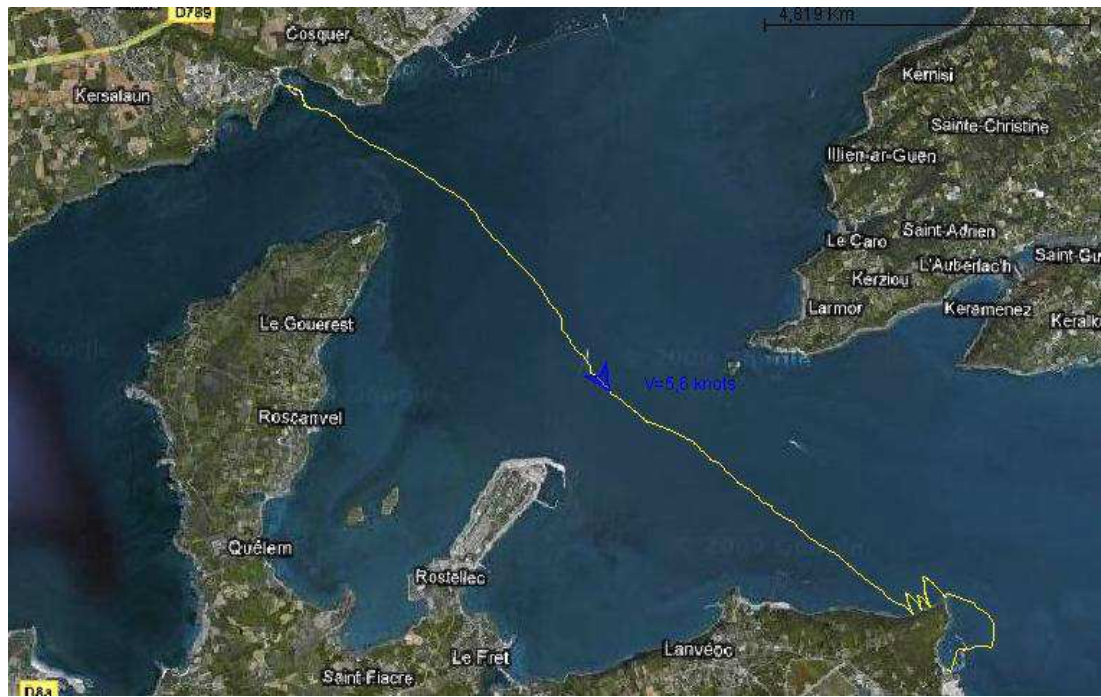






5 Sailboat robotics





2006 Volvo Ocean Race
2007 Aberystwyth Race
2010 Transatlantic Race
2011 Transatlantic Race
Live Tracking
Related Competitions
Photo Gallery
Videos
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Publications
Links
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France is the start line.
If you are not seeing any tracks on the map try reloading the page, sometimes they don't appear. Alternatively you can download the map for viewing in [google earth](#).

Boat	Team	Status	Latitude	Longitude	Time	Time Sailing
Green	Breizh Spirit	Started	46.435	-5.3907	2011-09-24 19:49:47	197.8
Green	Breizh Spirit	Started	46.435	-5.3907	2011-09-24 19:49:47	197.8

Breizh Spirit 2011-09-16 17:44:29
Spot Message

Breizh Spirit 2011-09-24 19:49:47
Spot Message

Données cartographiques ©2011 Google, Tele Atlas







6 Vaimos

Collaboration ENSTA/IFREMER



Vaimos à la WRSC (ENSTA-IFREMER-Ecole Navale).

$$\left\{ \begin{array}{lcl} \dot{x} & = & v \cos \theta + p_1 a \cos \psi \\ \dot{y} & = & v \sin \theta + p_1 a \sin \psi \\ \dot{\theta} & = & \omega \\ \dot{v} & = & \frac{f_s \sin \delta_s - f_r \sin u_1 - p_2 v^2}{p_9} \\ \dot{\omega} & = & \frac{f_s (p_6 - p_7 \cos \delta_s) - p_8 f_r \cos u_1 - p_3 \omega}{p_{10}} \\ f_s & = & p_4 a \sin (\theta - \psi + \delta_s) \\ f_r & = & p_5 v \sin u_1 \\ \sigma & = & \cos (\theta - \psi) + \cos (u_2) \\ \delta_s & = & \left\{ \begin{array}{ll} \pi - \theta + \psi & \text{si } \sigma \leq 0 \\ \text{sign} (\sin (\theta - \psi)) . u_2 & \text{sinon.} \end{array} \right. \end{array} \right.$$

The robot satisfies a state equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) .$$

With the controller $\mathbf{u} = \mathbf{g}(\mathbf{x})$, the robot satisfies

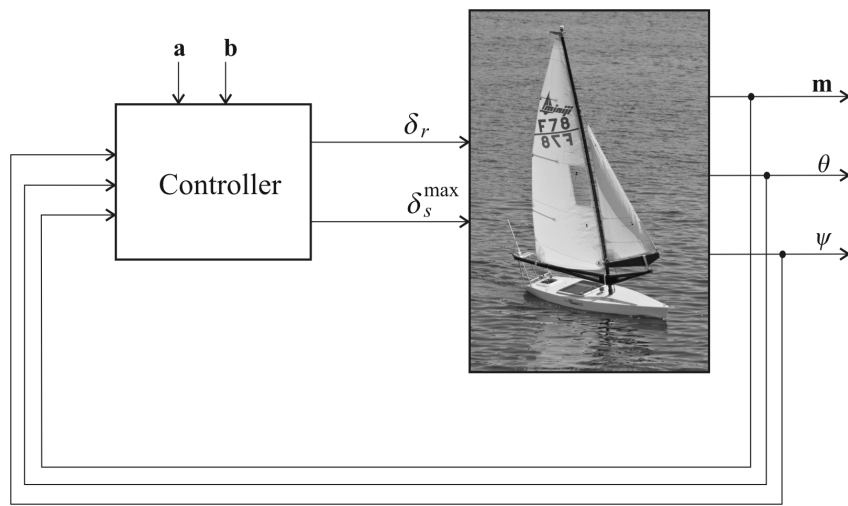
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) .$$

With all uncertainties, the robot satisfies.

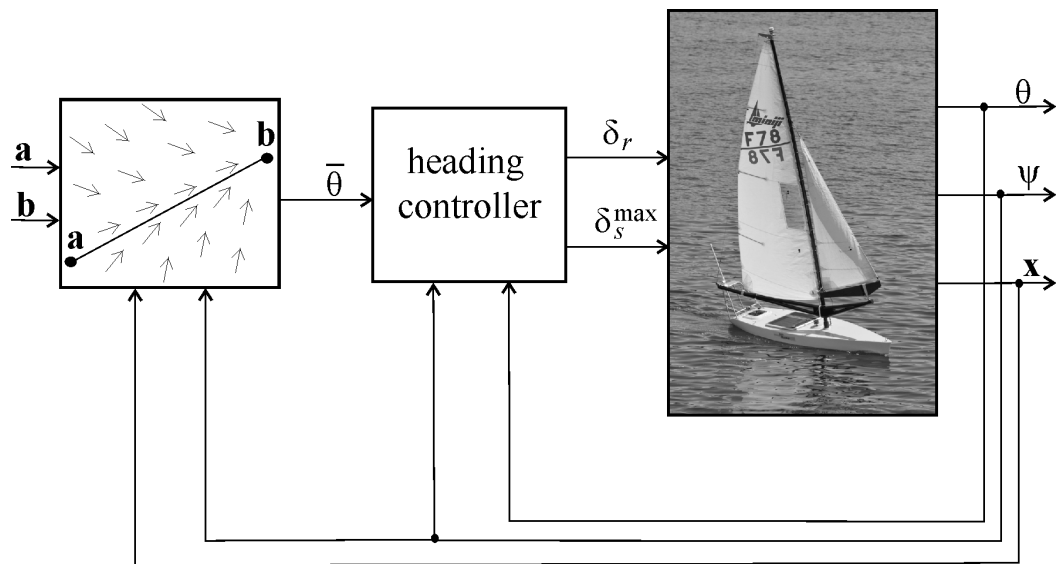
$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$$

which is *a differential inclusion*.

7 Line following



Controller of a sailboat robot

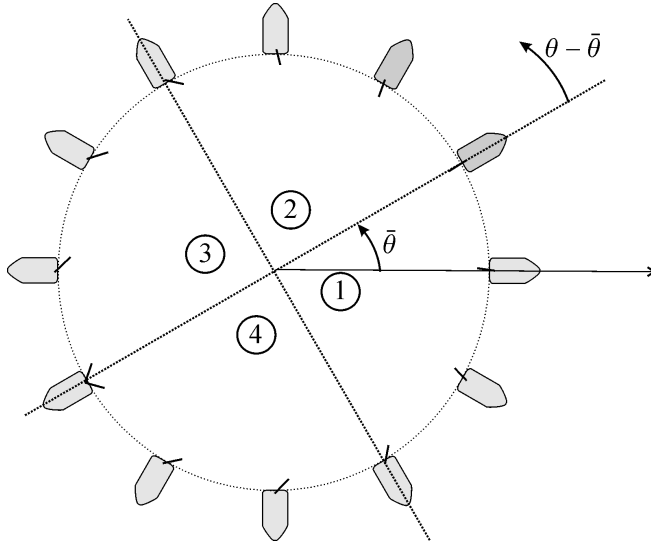


Heading controller

$$\begin{cases} \delta_r &= \frac{\delta_r^{\max}}{\pi} \cdot \text{atan}\left(\tan \frac{\theta - \bar{\theta}}{2}\right) \\ \delta_s^{\max} &= \frac{\pi}{2} \cdot \left(\frac{\cos(\psi - \bar{\theta}) + 1}{2} \right) . \end{cases}$$

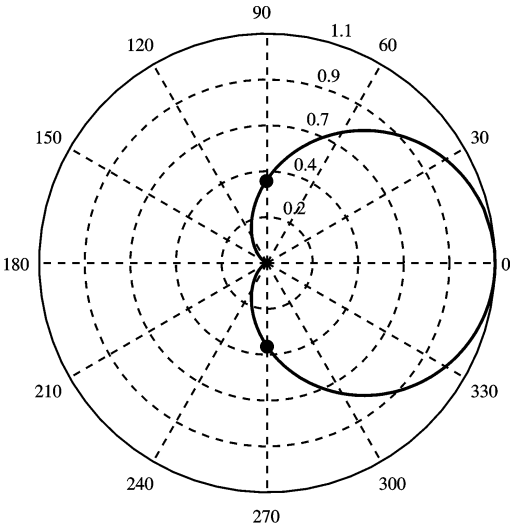
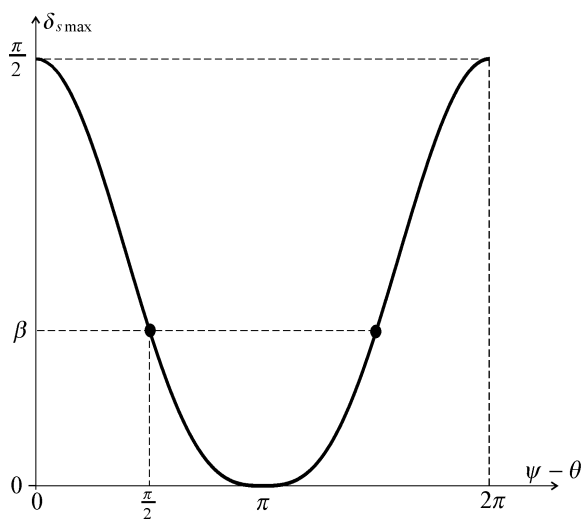
Rudder

$$\left\{ \begin{array}{l} \delta_r = \frac{\delta_r^{\max}}{\pi} \cdot \text{atan}\left(\tan \frac{\theta - \bar{\theta}}{2}\right) \end{array} \right.$$

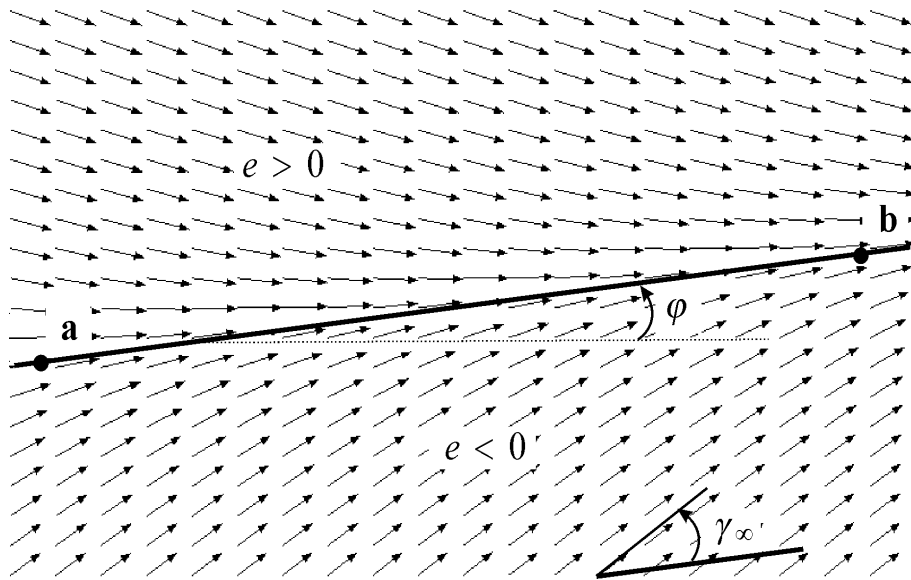


Sail

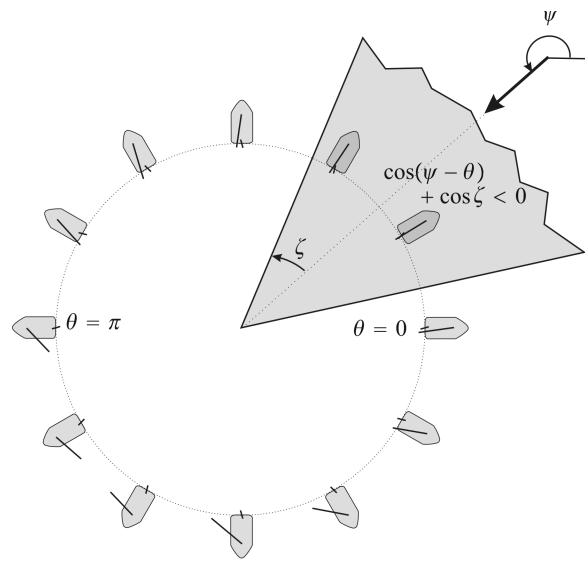
$$\delta_s^{\max} = \frac{\pi}{2} \cdot \left(\frac{\cos \left(\psi - \bar{\theta} \right) + 1}{2} \right)$$



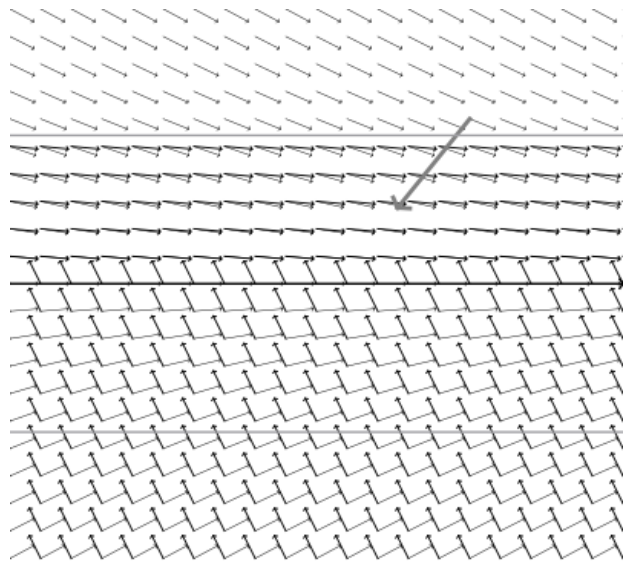
7.1 Vector field



Nominal vector field: $\theta^* = \varphi - \frac{1}{2} \cdot \text{atan}\left(\frac{e}{r}\right)$



A course θ^* may be unfeasible



Keep close hauled strategy.

7.2 Controller

Controlleur : in: $\mathbf{m}, \theta, \psi, \mathbf{a}, \mathbf{b}$; out: $\delta_r, \delta_s^{\max}$; inout: q

$$1 \quad e = \frac{\det(\mathbf{b}-\mathbf{a}, \mathbf{m}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|}$$

$$2 \quad \text{if } |e| > \frac{r}{2} \text{ then } q = \text{sign}(e)$$

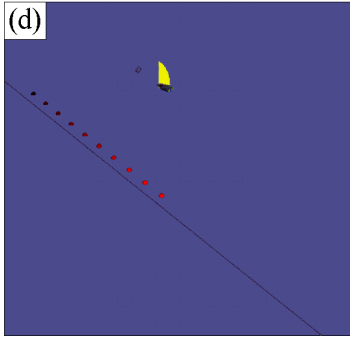
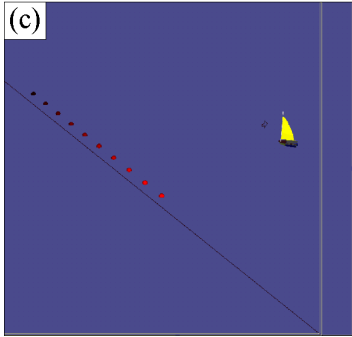
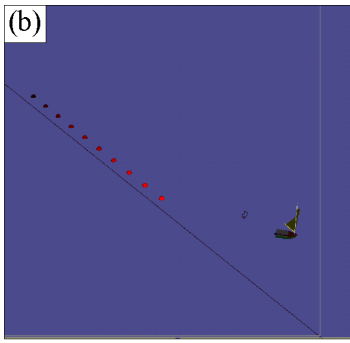
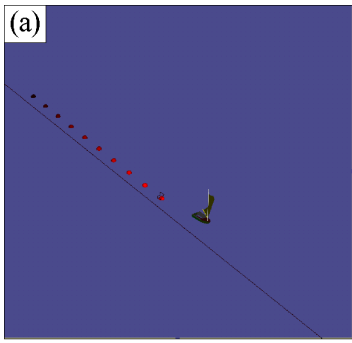
$$3 \quad \bar{\theta} = \text{atan2}(\mathbf{b} - \mathbf{a}) - \frac{1}{2} \cdot \text{atan}\left(\frac{e}{r}\right)$$

$$4 \quad \text{if } \cos(\psi - \bar{\theta}) + \cos \zeta < 0 \text{ then } \bar{\theta} = \pi + \psi - q \cdot \zeta.$$

$$5 \quad \delta_r = \frac{\delta_r^{\max}}{\pi} \cdot \text{atan}\left(\tan \frac{\theta - \bar{\theta}}{2}\right)$$

$$6 \quad \delta_s^{\max} = \frac{\pi}{2} \cdot \left(\frac{\cos(\psi - \bar{\theta}) + 1}{2} \right).$$

8 Validation by simulation



9 Theoretical validation

*Jaulin, Le Bars (2012). An interval approach for stability analysis; Application to sailboat robotics. IEEE TRO.

When the wind is known, the sailboat with the heading controller is described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) .$$

The system

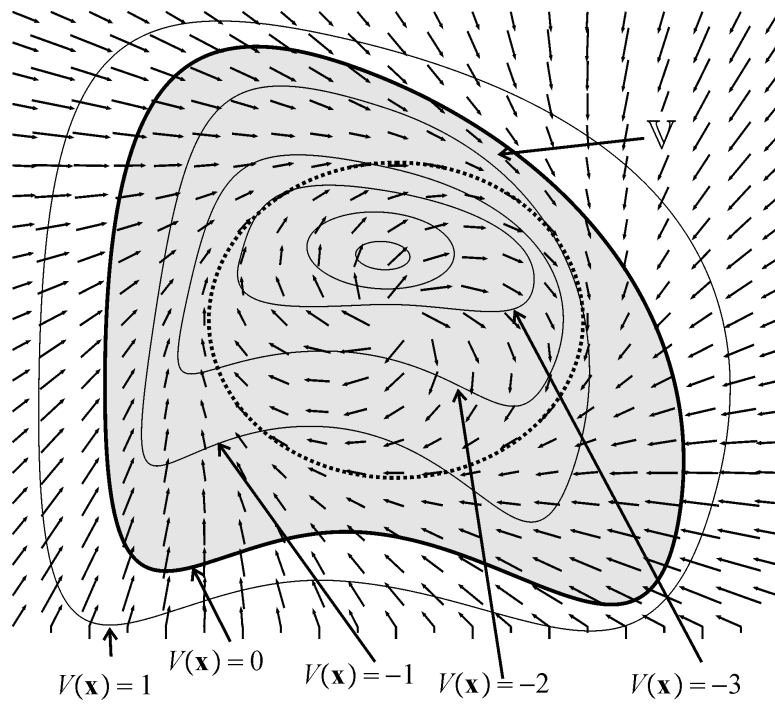
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is Lyapunov-stable (1892) if there exists $V(\mathbf{x}) \geq 0$ such that

$$\begin{aligned}\dot{V}(\mathbf{x}) &< 0 \text{ if } \mathbf{x} \neq \mathbf{0}, \\ V(\mathbf{x}) &= 0 \text{ iff } \mathbf{x} = \mathbf{0}.\end{aligned}$$

Definition. Consider a differentiable function $V(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$. The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V -stable if

$$\left(V(\mathbf{x}) \geq 0 \Rightarrow \dot{V}(\mathbf{x}) \leq \varepsilon < 0 \right).$$



Theorem. If the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is V -stable then

- (i) $\forall \mathbf{x}(0), \exists t \geq 0$ such that $V(\mathbf{x}(t)) < 0$
- (ii) if $V(\mathbf{x}(t)) < 0$ then $\forall \tau > 0, V(\mathbf{x}(t + \tau)) < 0$.

Now,

$$\begin{aligned} & \left(V(\mathbf{x}) \geq 0 \Rightarrow \dot{V}(\mathbf{x}) < 0 \right) \\ \Leftrightarrow & \left(V(\mathbf{x}) \geq 0 \Rightarrow \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0 \right) \\ \Leftrightarrow & \forall \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) < 0 \text{ or } V(\mathbf{x}) < 0 \\ \Leftrightarrow & \neg \left(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \geq 0 \text{ and } V(\mathbf{x}) \geq 0 \right) \end{aligned}$$

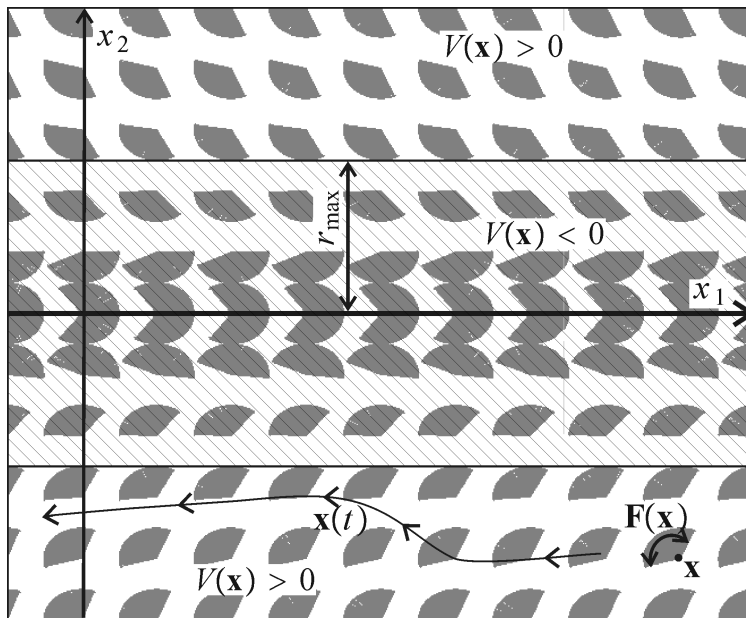
Theorem. We have

$$\begin{cases} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{cases} \text{ inconsistent} \Leftrightarrow \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is } V\text{-stable.}$$

Interval method could easily prove the V -stability.

Theorem. We have

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}) \cdot \mathbf{a} \geq 0 \\ \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ V(\mathbf{x}) \geq 0 \end{array} \right. \text{ inconsistent } \Leftrightarrow \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}) \text{ is } V\text{-stable}$$

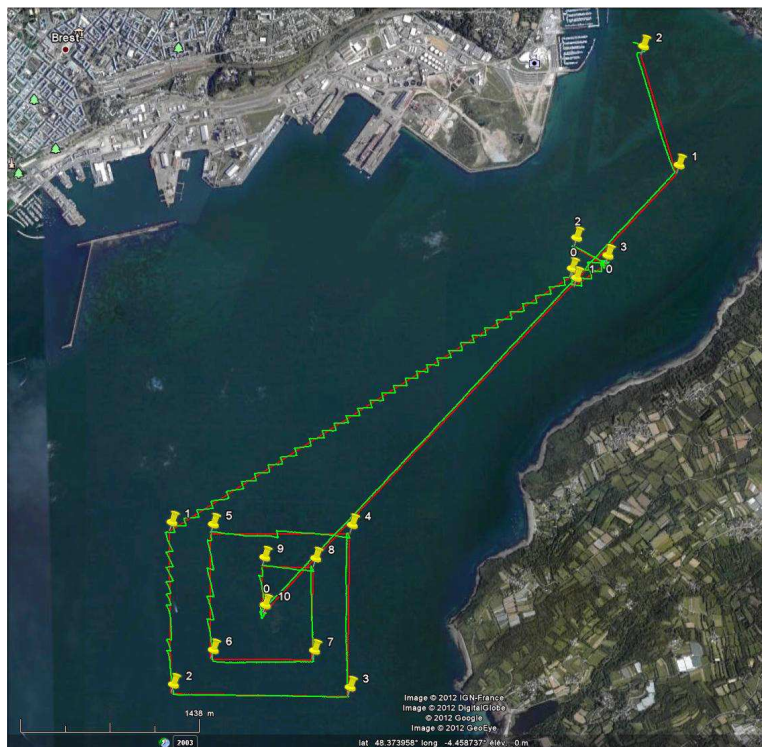


Differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ for the sailboat.

$$V(x) = x_2^2 - r_{\max}^2.$$

10 Experimental validation

Collaboration ENSTA-Ifremer. Fabrice Le Bars, Olivier Ménage, Patrick Rousseau, . . .



Rade de Brest

Brest-Douarnenez. January 17, 2012, 8am

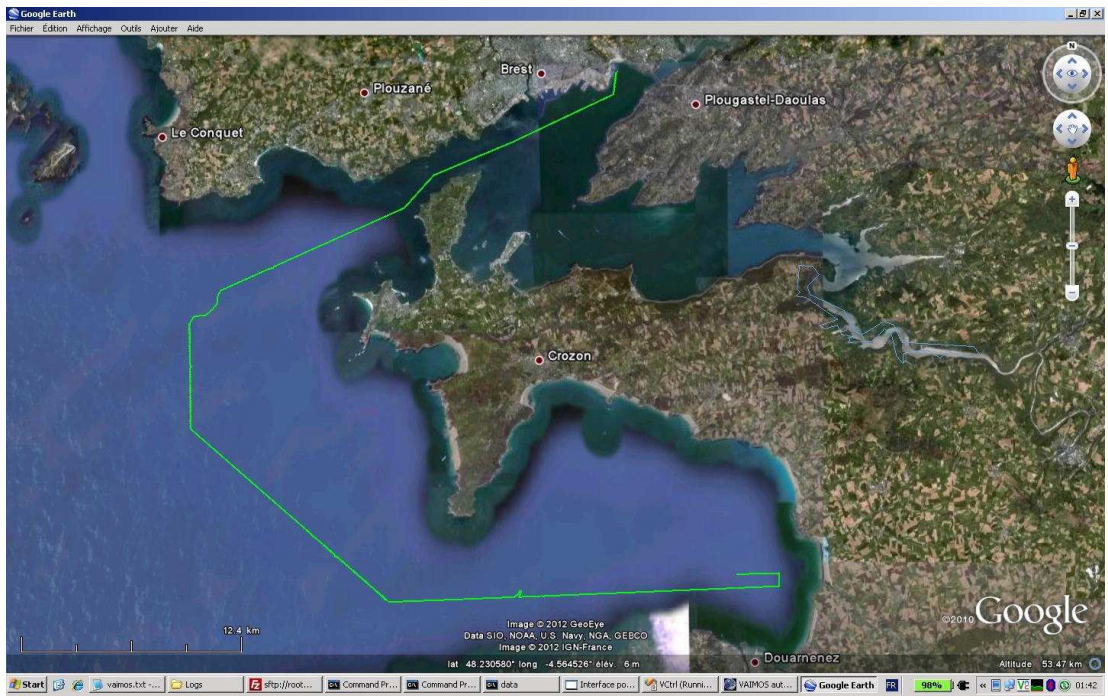


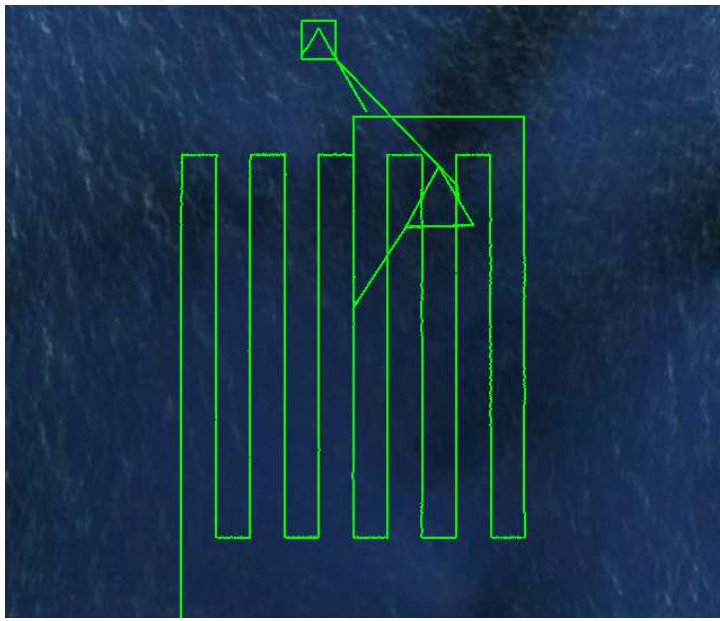












Middle of Atlantic ocean, 350 km made by Vaimos in
53h, September 6-9, 2012.

Consequence.

It is possible for a sailboat robot to navigate inside a corridor.

Essential, to create circulation rules when robot swarms are considered.

Essential to determine who has to pay in case of accident.