A new wrapper for a reliable resolution of underdetermined nonlinear equations

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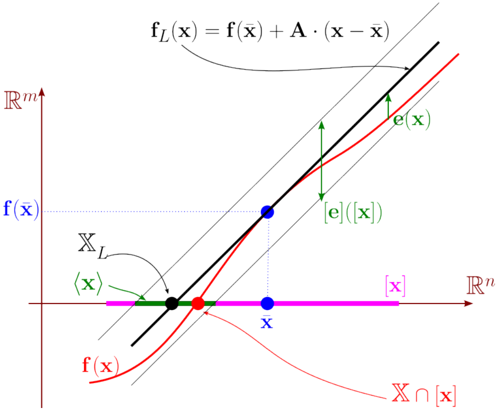
1. First order enclosure

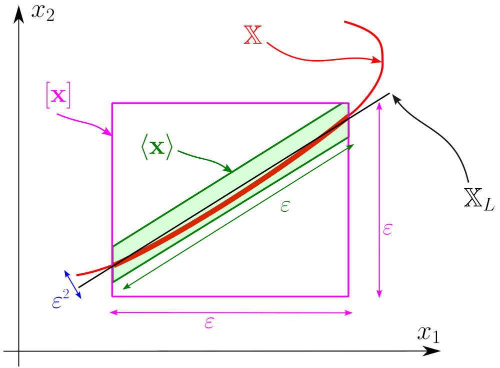
We want to characterize the set

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$, $m < n$.



Principle of the first order enclosure: $w([\mathbf{e}]([\mathbf{x}])) = o(w([\mathbf{x}]))$



$$\mathrm{Vol}(\langle\mathbf{x}\rangle) = O(\varepsilon^n \cdot \varepsilon^{k(n-m)}) = O(\varepsilon^3), k = 1 \text{ whereas}$$
$$\mathrm{Vol}([\mathbf{x}]) = O(\varepsilon^n) = O(\varepsilon^2)$$

Polyhedron approximation of $\mathbb{X} \cap [\mathbf{x}]$

Proposition

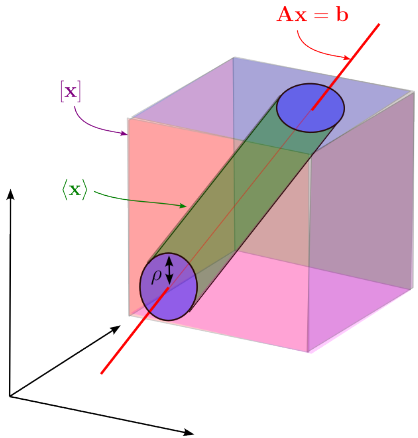
$$\begin{cases} \mathbf{f}(\mathbf{x}) = \mathbf{0} \\ \mathbf{x} \in [\mathbf{x}] \end{cases} \Rightarrow \begin{cases} \mathbf{A} \cdot \mathbf{x} \in [\mathbf{b}] \\ \mathbf{A} = \frac{d\mathbf{f}}{d\mathbf{x}}(\bar{\mathbf{x}}) \\ [\mathbf{b}] = \mathbf{A}\bar{\mathbf{x}} - \mathbf{f}(\bar{\mathbf{x}}) + [\mathbf{e}] \\ [\mathbf{e}] = \left(\mathbf{A} - \frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])\right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \end{cases}$$

2. Gutters

The *gutter* associated to $[\mathbf{x}] \subset \mathbb{R}^n$, a matrix $\mathbf{A}$, a vector $\mathbf{b}$ and the radius ρ is

$$\begin{array}{rcl} \langle \mathbf{x} \rangle & = & \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle \\ & = & \{\mathbf{x} \in [\mathbf{x}], \exists \mathbf{p}, \mathbf{A}\mathbf{p} = \mathbf{b} \text{ and } \|\mathbf{x} - \mathbf{p}\| < \rho\}. \end{array}$$

The affine space $\mathbf{A}\mathbf{p} = \mathbf{b}$ is called a *flat*.


$\mathbf{Ax} = \mathbf{b}$
$[\mathbf{x}]$
$\langle\mathbf{x}\rangle$
ρ

Motivation for using gutters is:

- The box $[\mathbf{x}]$ in the gutter allows to build a nonoverlapping covering of $\mathbb{X}$.
- A gutter can easily be bisected
- The axis-aligned projection is easy
- Gutters can easily be intersected
- A first order approximation is possible

Projection of a gutter

The projection of $\langle \mathbf{x} \rangle = \langle [\mathbf{x}_1] \times [\mathbf{x}_2], (\mathbf{A}_1, \mathbf{A}_2), \mathbf{b}, \rho \rangle$ on the $\mathbf{x}_1$-space is defined by:

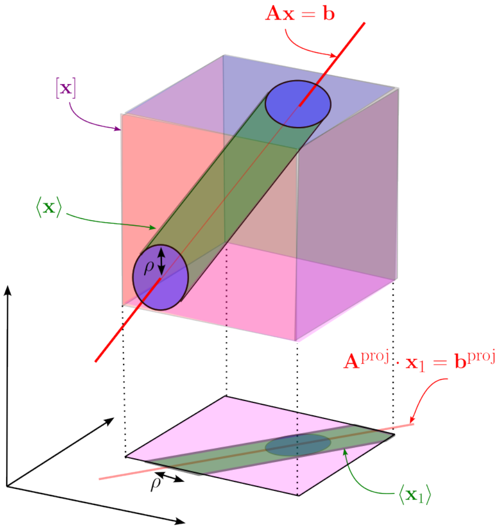
$$\mathsf{proj}_{1:m} \langle \mathbf{x} \rangle =< [\mathbf{x}_1], \mathbf{A}^{\mathsf{proj}}, \mathbf{b}^{\mathsf{proj}}, \rho >$$

where

$$\begin{array}{rcl} \mathbf{A}^{\mathsf{proj}} & = & \left(\mathbf{A}_1^{-1}\mathbf{A}_2\right)^{\perp} \\ \mathbf{b}^{\mathsf{proj}} & = & \mathbf{A}^{\mathsf{proj}}\mathbf{A}_1^{-1}\mathbf{b} \end{array}$$

Proposition.

$$\mathbf{x} \in \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle \Rightarrow \mathsf{proj}_{1:m}\mathbf{x} \in \mathsf{proj}_{1:m} \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$


$\mathbf{A}\mathbf{x} = \mathbf{b}$
$[\mathbf{x}]$
$\langle \mathbf{x} \rangle$
ρ
$\mathbf{A}^{\mathrm{proj}} \cdot \mathbf{x}_1 = \mathbf{b}^{\mathrm{proj}}$
ρ
$\langle \mathbf{x}_1 \rangle$

Intersection of two gutters

The intersection between $\langle \mathbf{x}_1 \rangle$ and $\langle \mathbf{x}_2 \rangle$ is defined by

$$\langle [\mathbf{x}]_1, \mathbf{A}_1, \mathbf{b}_1, \rho_1 \rangle \cap \langle [\mathbf{x}]_2, \mathbf{A}_2, \mathbf{b}_2, \rho_2 \rangle = \langle [\mathbf{x}]_3, \mathbf{A}_3, \mathbf{b}_3, \rho_3 \rangle$$

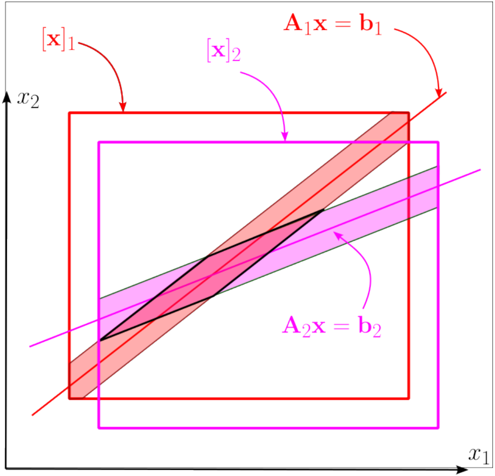
where

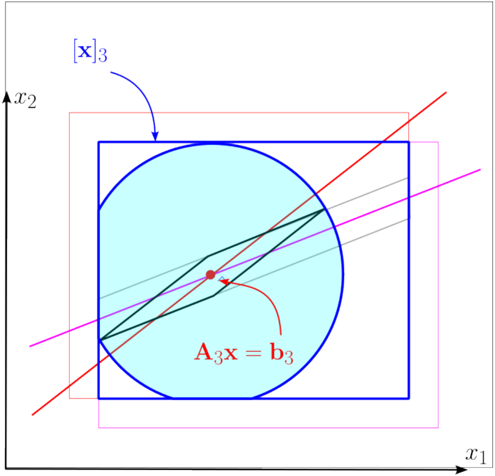
$$\begin{array}{rcc} [\mathbf{x}]_3 & = & [\mathbf{x}]_1 \cap [\mathbf{x}]_2 \\ \mathbf{A}_3 & = & \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \\ \mathbf{b}_3 & = & \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \\ \rho_3 & = & \frac{1}{\sin\theta} \sqrt{\rho_b^2 + \rho_a^2 + 2\rho_a\rho_b \cdot \cos\theta} \end{array}$$

where θ is the principle angle between $\mathrm{span}(\mathbf{A}_1)$ and $\mathrm{span}(\mathbf{A}_2)$.

Proposition

$$\mathbf{x} \in \langle \mathbf{x}_1 \rangle \text{ and } \mathbf{x} \in \langle \mathbf{x}_2 \rangle \Rightarrow \mathbf{x} \in \langle \mathbf{x}_1 \rangle \cap \langle \mathbf{x}_2 \rangle .$$


$[\mathbf{x}]_1$
$[\mathbf{x}]_2$
$\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$
x_2
$\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$
x_1


$[\mathbf{x}]_3$
x_2
x_1
$\mathbf{A}_3\mathbf{x} = \mathbf{b}_3$

3. Gutter contractors

Consider again

$$\mathbb{X} = \{\mathbf{x} \in [\mathbf{x}] | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$, $m < n$.

A *gutter contractor* $\mathscr{G}([\mathbf{x}])$ associated to $\mathbb{X}$ is an operator

$$\mathscr{G} : [\mathbf{x}] \rightarrow \langle \mathbf{x} \rangle = \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$

which satisfies

$$\mathbb{X} \cap [\mathbf{x}] \subset \langle \mathbf{x} \rangle .$$

Moreover, $\mathscr{G}$ is said to have an order k if for all nested sequence of boxes converging to $\mathbf{x} \in \mathbb{X}$,

$$\frac{h(\langle \mathbf{x} \rangle, \mathbb{X} \cap [\mathbf{x}])}{\mathrm{rad}([\mathbf{x}])^k} \rightarrow 0.$$

Proposition. For $\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$, the operator

$$\mathscr{G} : [\mathbf{x}] \to \langle [\mathbf{x}], \mathbf{A}, \mathbf{b}, \rho \rangle$$

where

$$\begin{array}{rcl} \mathbf{b} & = & \mathbf{A}\bar{\mathbf{x}} - \mathbf{f}(\bar{\mathbf{x}}) \\ \mathbf{A} & = & \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) \\ \rho & = & \|\mathbf{A}^\mathsf{T}\left(\mathbf{A}\mathbf{A}^\mathsf{T}\right)^{-1}\| \cdot \mathrm{ub}\|[\mathbf{e}]\| \\ [\mathbf{e}] & = & \left(\mathbf{A} - \frac{d\mathbf{f}}{d\mathbf{x}}([\mathbf{x}])\right) \cdot ([\mathbf{x}] - \bar{\mathbf{x}}) \end{array}$$

is a gutter contractor of order 1.

Algorithm with gutters

Initialization: $\mathscr{L} = \{[\mathbf{x}]\}$; $\mathbb{X}^+ = \{\}$.
Resolution.

- **Contraction step**. Replace each $[\mathbf{x}] \in \mathscr{L}$, by the smallest box enclosing the gutter $\langle \mathbf{x} \rangle = \mathscr{G}([\mathbf{x}])$.
- **Bisection step**. For each $[\mathbf{x}] \in \mathscr{L}$, if Vol$(\langle \mathbf{x} \rangle) < \varepsilon^n$ then push $\langle \mathbf{x} \rangle$ on $\mathbb{X}^+$, otherwise, bisect $[\mathbf{x}]$ and push the two resulting boxes in $\mathscr{L}$.

4. Test-cases

Test-case 1 : Turkulov system

Consider the linear time-delay system given by

$$\ddot{y} + 2\dot{y}(t - x_1) + y(t - x_2) = 0$$

where x_1, x_2 are two parameters. Its characteristic function is

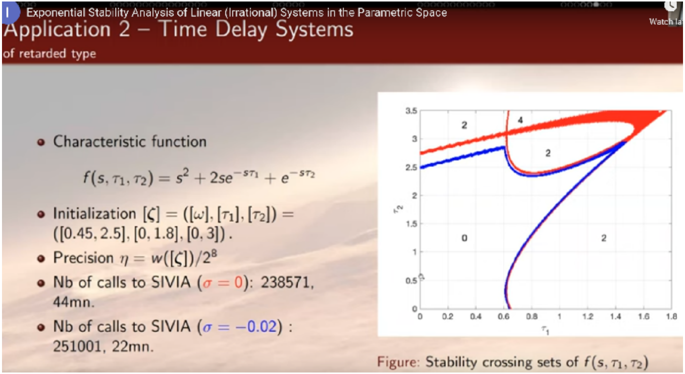
$$\theta(\mathbf{x}, s) = s^2 + 2se^{-sx_1} + e^{-sx_2}.$$

The location of the roots for $\theta(x_1, x_2, jx_3)$, provides a good understanding of the stability of the system.

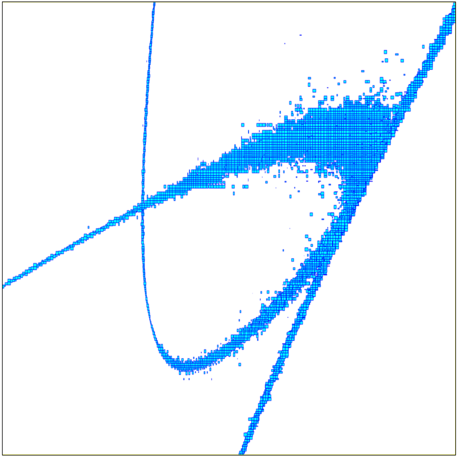
Now

$$
\begin{aligned}
& \theta(x_1, x_2, jx_3) = 0 \\
\Leftrightarrow \quad & \mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_3^2 + 2x_3 \sin(x_3 x_1) + \cos(x_3 x_2) \\ 2x_3 \cos(x_3 x_1) - \sin(x_3 x_2) \end{pmatrix} = \mathbf{0}
\end{aligned}
$$

With $[x_1] = [0, 2.5]$, $[x_2] = [1, 4]$, $[x_3] = [0, 10]$, a Matlab implementation, with a forward-backward contractor, and $\varepsilon = 2^{-8}$, yields:

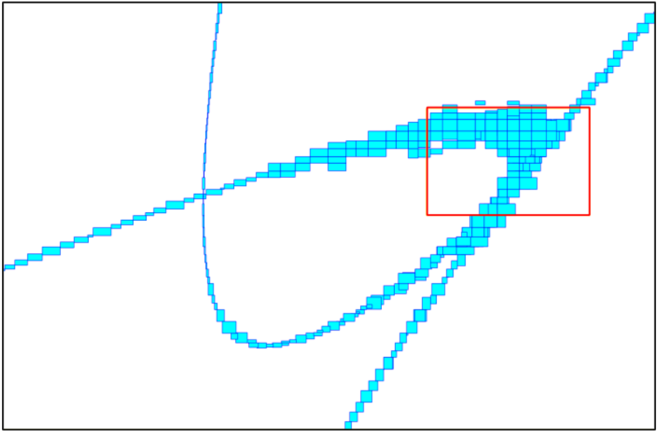


https://youtu.be/DaR2NZZIV10?t=2453

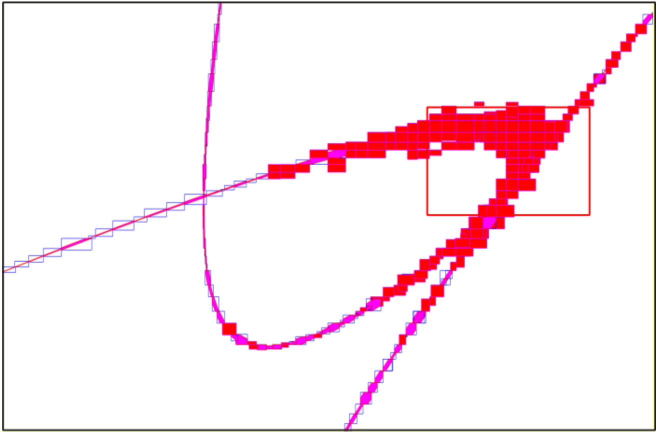


$\varepsilon = 2^{-8}$, Codac generated 43173 boxes.

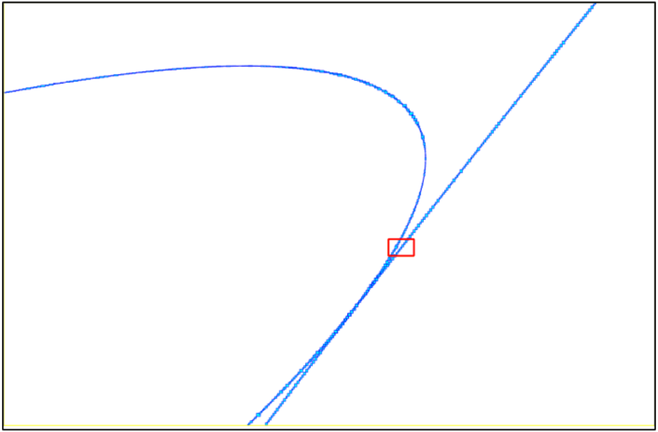
We still have a *Clustering effect*



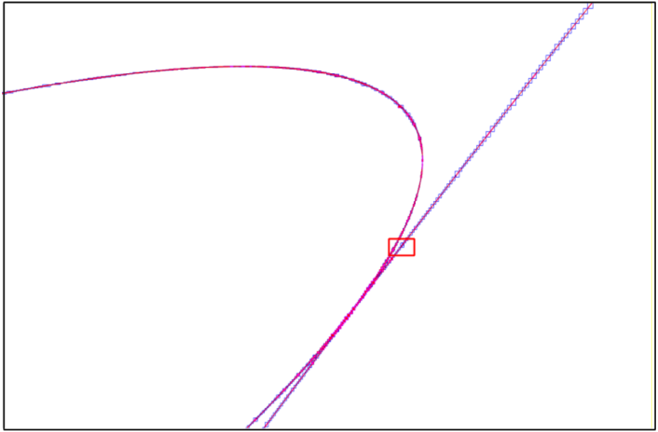
(a) Asymptotically minimal contractor



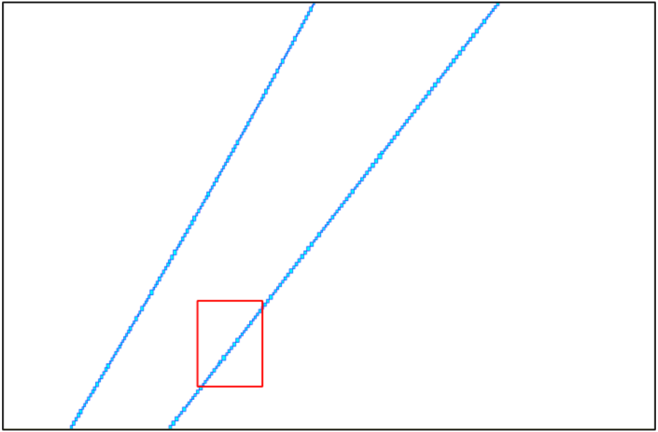
(a) Gutter contractor



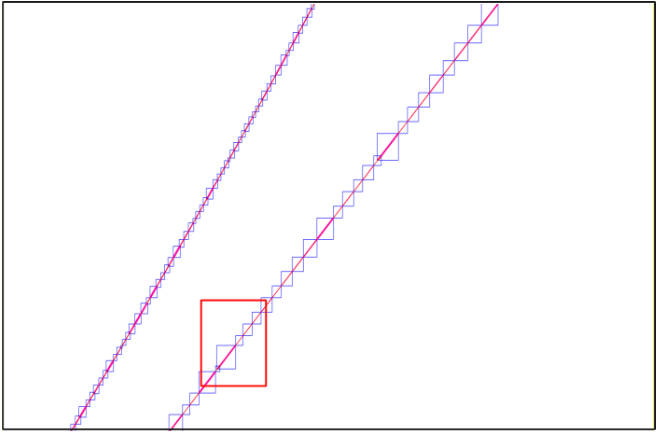
(b) Asymptotically minimal contractor



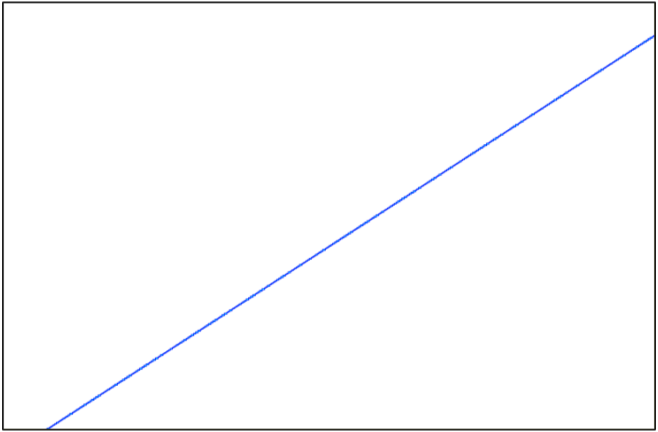
(b) Gutter contractor



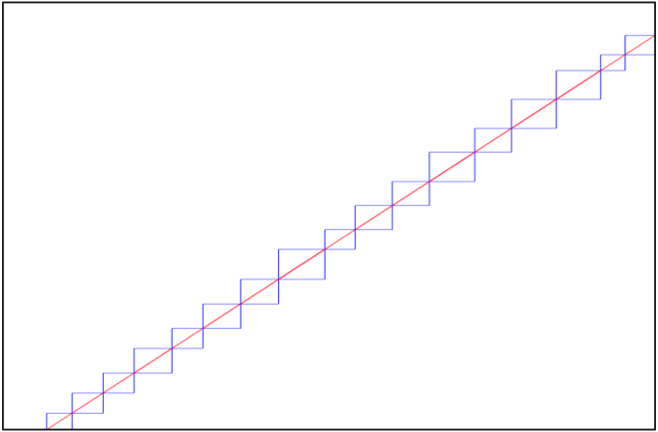
(c) Asymptotically minimal contractor



(c) Gutter contractor



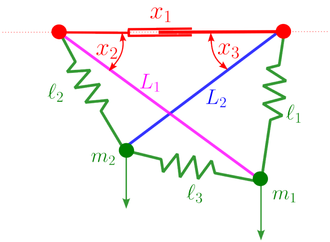
(d) Asymptotically minimal contractor



(d) Gutter contractor

	Case (a)	Case (b)	Case (c)	Case (d)
$[\mathbf{x}]$	$\begin{pmatrix} [0,2] \\ [2,4] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.3,1.8] \\ [3,3.5] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.595,1.615] \\ [3.2,3.22] \\ [0,10] \end{pmatrix}$	$\begin{pmatrix} [1.601,1.603] \\ [3.202,3.206] \\ [0,10] \end{pmatrix}$
ε	2^{-4}	2^{-8}	2^{-12}	2^{-16}
$T_1(s)$	0.51	0.84	0.11	0.19
$T_2(s)$	0.86	1.46	0.17	0.05
V_1	0.092	$2.05 \cdot 10^{-3}$	$3.06 \cdot 10^{-6}$	$4.18 \cdot 10^{-7}$
V_2	0.02	$1.93 \cdot 10^{-3}$	$7.2 \cdot 10^{-7}$	$5.78 \cdot 10^{-9}$
$\frac{V_1}{V_2}$	3.8	4.6	15.3	72.4

Test-case 2 : Tensegrity mechanism



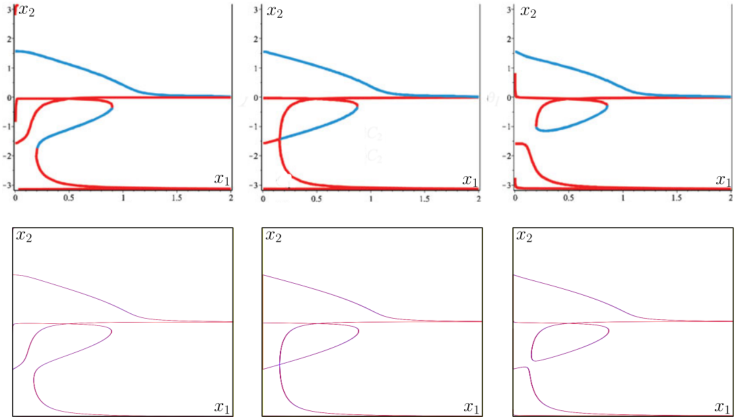
The length x_1 can be tuned.

At the equilibrium, the potential energy:

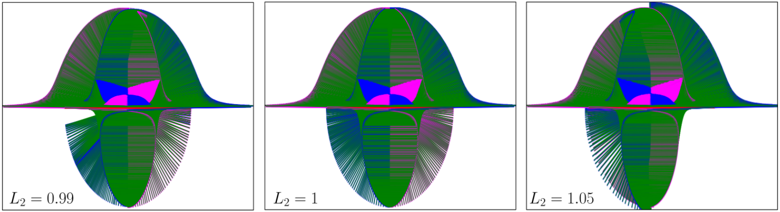
$$\begin{aligned} U &= \frac{k}{2}\left(\ell_1^2+\ell_2^2+\ell_3^2\right) - m_1 L_1 \sin x_2 - m_2 L_2 \sin x_3 \\ &= \frac{k}{2}\left(3x_1^2 - 4L_1 x_1 \cos\theta_1\right) \\ &\quad +k\left(L_1^2+L_2^2+L_1L_2\cos(x_2+x_3) - 4L_2 x_1 \cos x_3\right) \\ &\quad -m_1 L_1 \sin x_2 - m_2 L_2 \sin x_3 \end{aligned}$$

should be stationary.

$$\begin{aligned}
kL_1 \sin(x_2 + x_3) - 2kx_1 \sin x_3 + m_2 \cos x_3 &= 0 \\
kL_2 \sin(x_2 + x_3) - 2kx_1 \sin x_2 + m_1 \cos x_2 &= 0
\end{aligned}$$



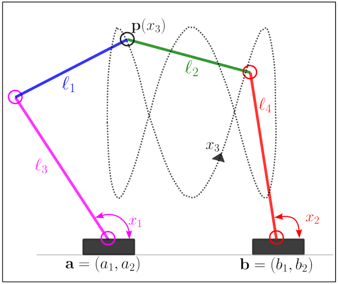
Top: symbolic approach [Wenger]; Bottom: with gutters
Left: $L_2 = 0.99$, Center: $L_2 = 1$; Right: $L_2 = 1.05$



In the workspace, representation of all configurations for the robot

Problem: How to prove the feasibility of a given tensegrity path?

Test-case 3: RP-5AH robot


$\mathbf{p}(x_3)$
ℓ_1
ℓ_2
ℓ_3
ℓ_4
x_3
x_1
x_2
$\mathbf{a} = (a_1, a_2)$
$\mathbf{b} = (b_1, b_2)$

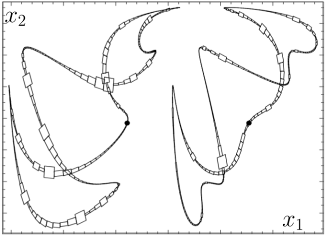
$\mathbf{p}$ should stay on a Lissajou curve, i.e.,

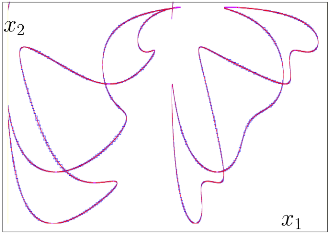
$$\mathbf{p} = \begin{pmatrix} \sin x_3 \\ \frac{3}{2} + \cos(3x_3) \end{pmatrix}$$

for some $x_3 \in [0, 2\pi]$.

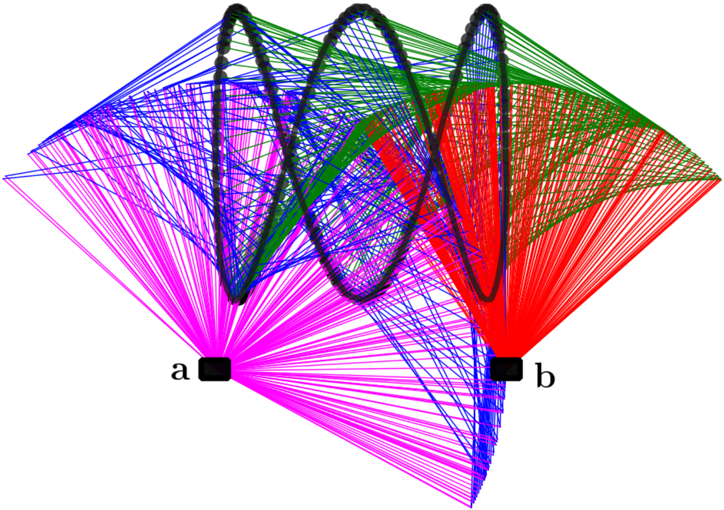
We have to solve

$$
\begin{aligned}
(\sin x_3 - \ell_3 \cos x_1 - a_1)^2 + (\tfrac{3}{2} + \cos(3x_3) - \ell_3 \sin x_1 + a_2)^2 - \ell_1^2 &= 0 \\
(\sin x_3 - \ell_4 \cos x_2 - b_1)^2 + (\tfrac{3}{2} + \cos(3x_3) - \ell_4 \sin x_2 - b_2)^2 - \ell_2^2 &= 0
\end{aligned}
$$





Left: continuation method, Right: with gutters


a
b

The code source of the three test-cases is based on the codac library [18] and can be found at:

https://www.ensta-bretagne.fr/jaulin/buche.html

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8. Test case 2: Tensegrity [2], [22]
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