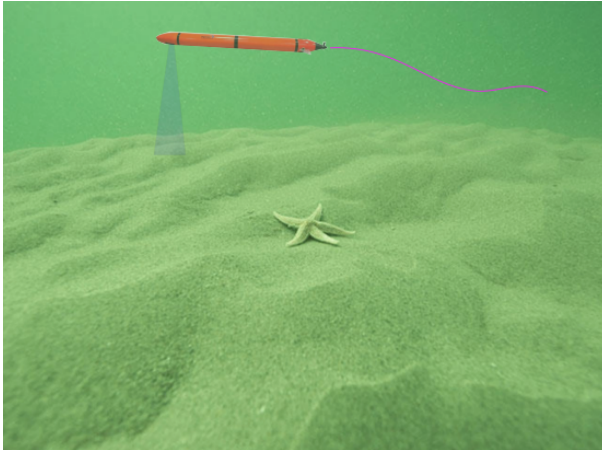


# Explore and return for an underwater robot in a minimalist environment with a few computation

Luc Jaulin, Quentin Brateau and Fabrice Le Bars

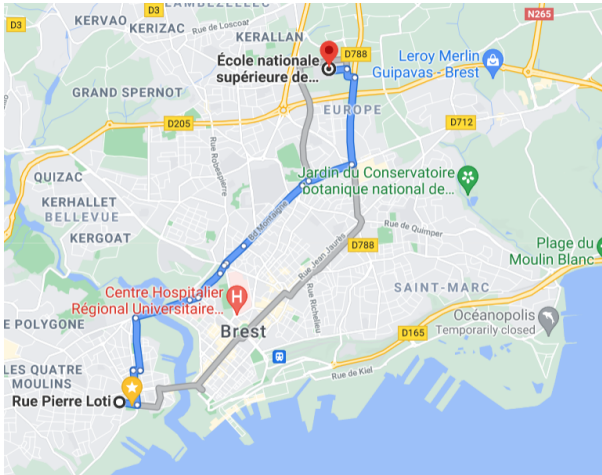


December 11, 2025, LIRMM, Montpellier



Explore and return in a minimalist environment

# Map-based navigation



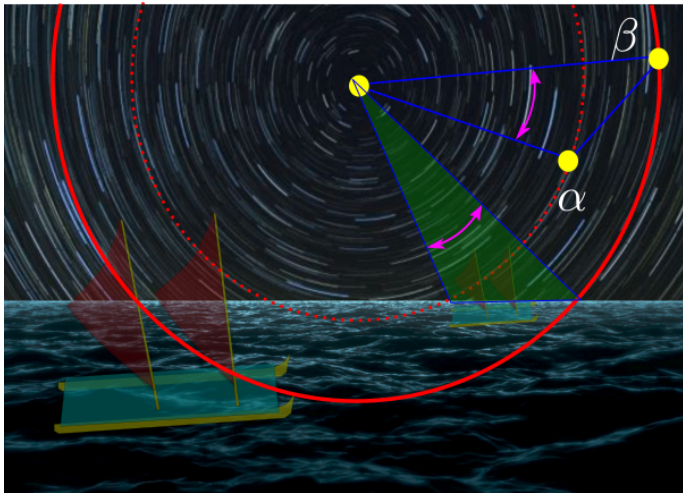
Modern navigation: high cost (computation, infrastructure)



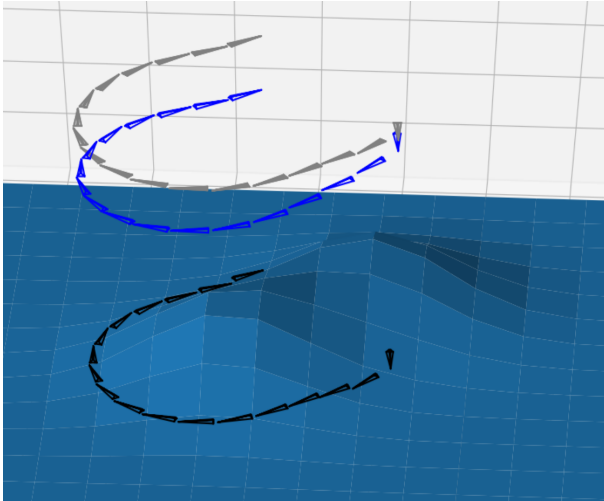
# 1. Route-based navigation

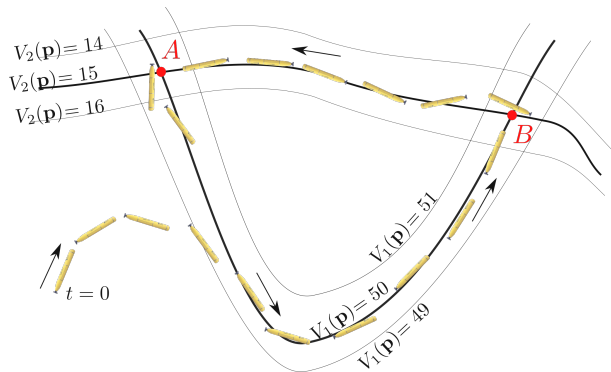
# Follow a route

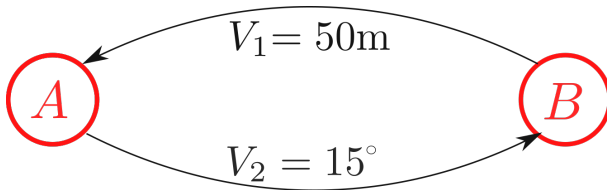
Given a function  $h: \mathbb{R}^2 \mapsto \mathbb{R}$ , a route is defined by  $h(\mathbf{p}) = 0$ .  
 $h$  could be the temperature, the radiation, the pressure, the altitude, the time shift between two periodic events.



The stars  $\alpha$  and  $\beta$  set simultaneously.  $h(\mathbf{p}) = t_\alpha - t_\beta$



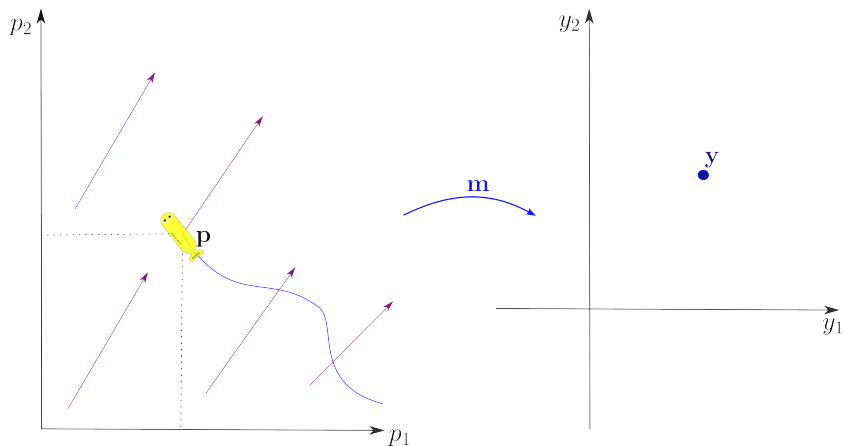




## 2. Field-based navigation



$$\begin{cases} \dot{p}_1 &= \cos q \\ \dot{p}_2 &= \sin q \\ \dot{q} &= u \end{cases} \quad \text{and} \quad \begin{cases} y_1 &= m_1(\mathbf{p}) \\ y_2 &= m_2(\mathbf{p}) \end{cases}$$



Or more generally

$$\begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{q}} \end{pmatrix} = \mathbf{f}(\mathbf{p}, \mathbf{q}, \mathbf{u})$$

where  $\mathbf{p} \in \mathbb{R}^2$  is the position.

The world is a 2D field

$$\mathbf{y} = \mathbf{m}(\mathbf{p}) \in \mathbb{R}^2$$

For instance, for magnetism based navigation

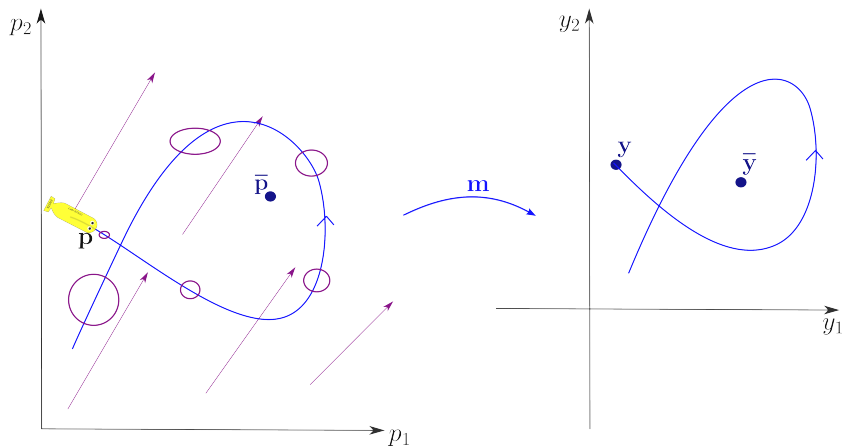
- $m_1$  is the inclination of the magnetic field
- $m_2$  is the amplitude of the magnetic field

Assume that we want to go to a position  $\bar{\mathbf{p}}$  where we know  $\bar{\mathbf{y}} = \mathbf{m}(\bar{\mathbf{p}})$ .

Think of the pigeon who returns to its loft using mag only.

We are able

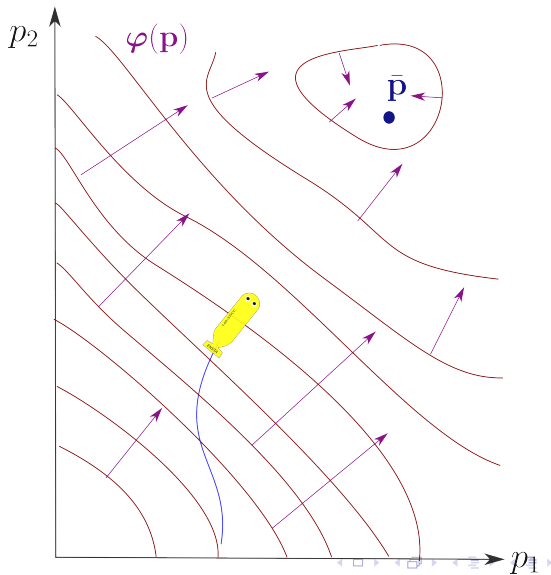
- to detect and prove the existence and unicity of a loop (topological degree)
- to prove the existence of  $\mathbf{p}$  such that  $\bar{\mathbf{y}} = \mathbf{m}(\mathbf{p})$
- Move in order to minimize  $\|\mathbf{m}(\mathbf{p}) - \bar{\mathbf{y}}\|^2$



**Strategies:** We build a clever field  $\phi(\mathbf{p})$  to follow.  
For instance, for a *reach home*, we take

$$\phi(\mathbf{p}) = \frac{\partial}{\partial \mathbf{p}} \|\mathbf{m}(\mathbf{p}) - \bar{\mathbf{y}}\|^2$$

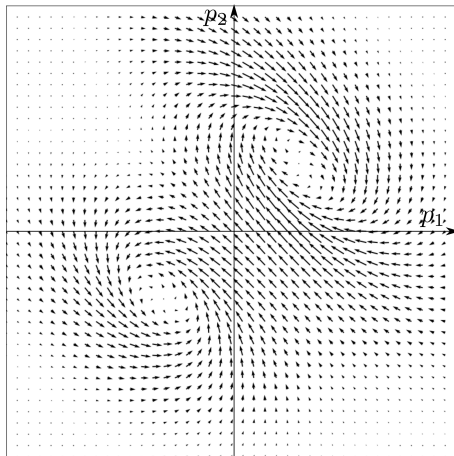




## Stable cycles

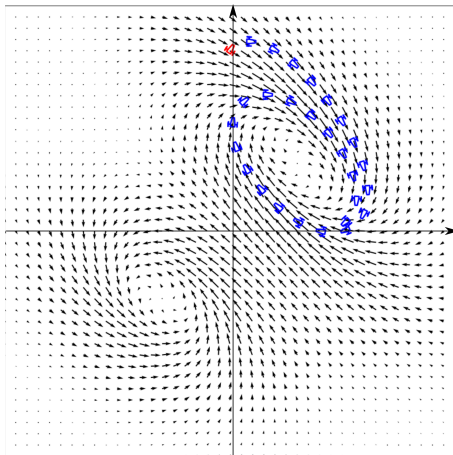
- Poincaré Bendixon (no chaos for in 2D)
- We find stable cycles, e.g.,

$$\varphi(\mathbf{p}) = \begin{pmatrix} m_2(\mathbf{p}) \\ -m_1(\mathbf{p}) \end{pmatrix} + \tanh(m_1(\mathbf{p}) - \bar{y}_1) \cdot \mathbf{m}(\mathbf{p})$$

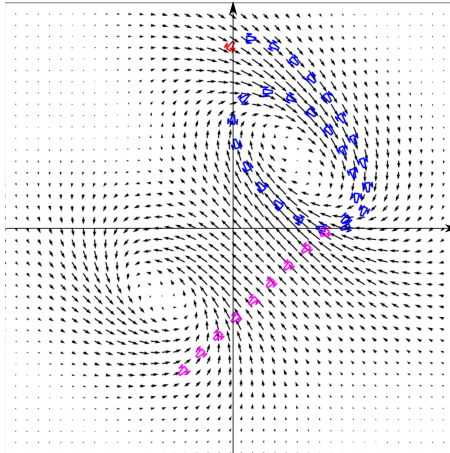


Field  $\varphi(\mathbf{p})$

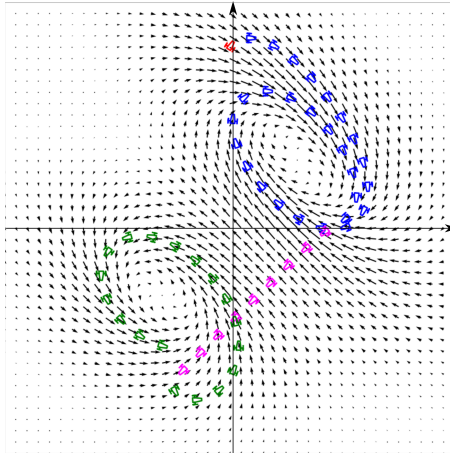
# Exploration



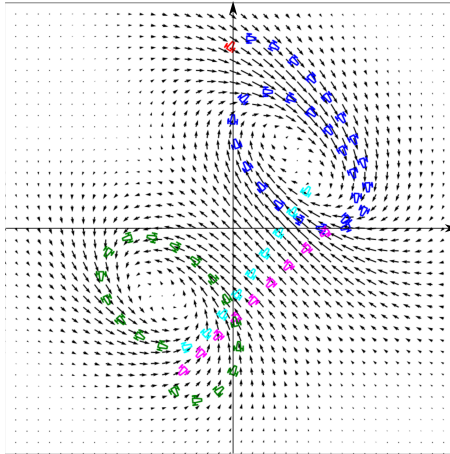
The robot follows the vector field and converges toward an attractor (Poincaré Bendixon)



For exploration, we go randomly to a given heading

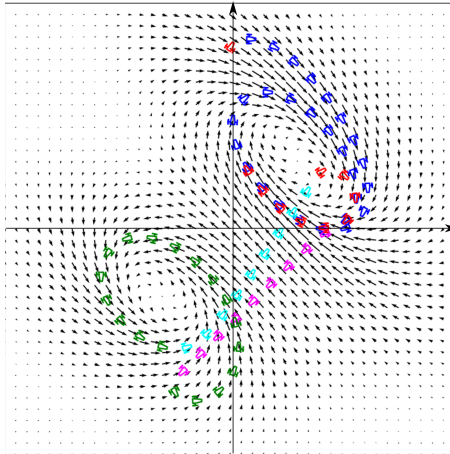


It searches and finds another stable cycle  $\mathbb{C}_2$ .



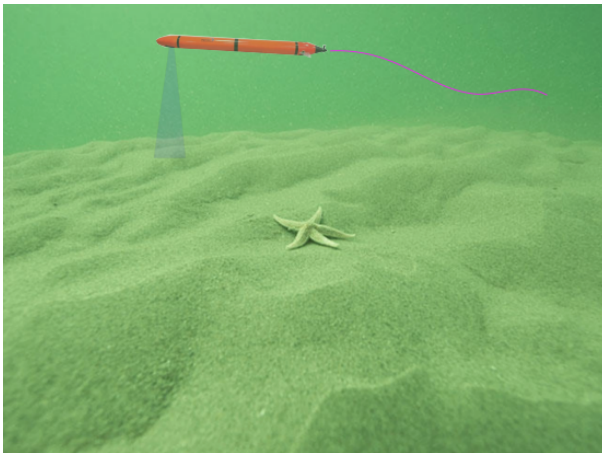
The robot comes back home



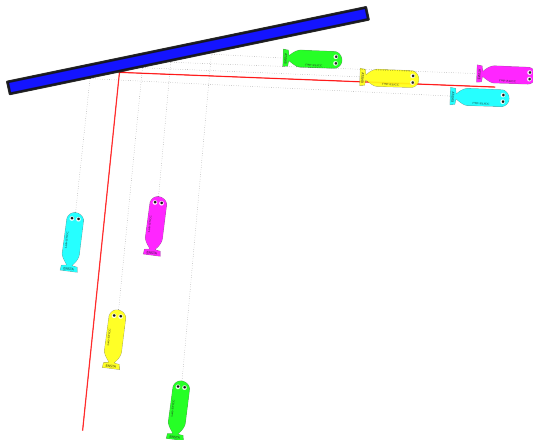


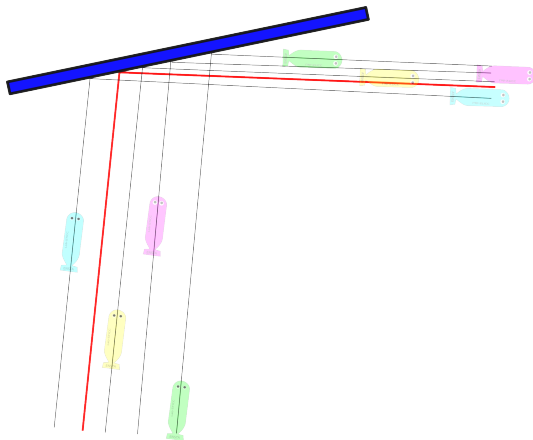
The robot follows its initial cycle. It was never lost.

### 3. Bounces-based navigation (phd of Q. Brateau)

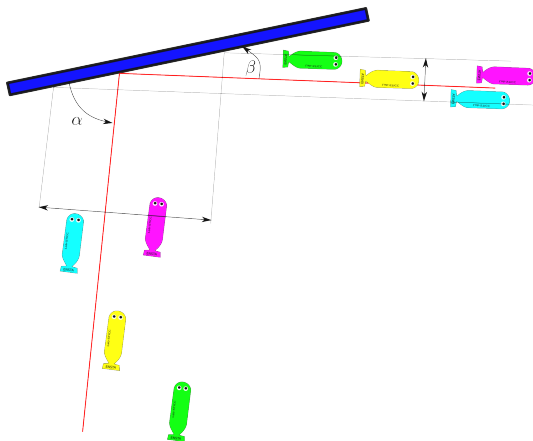


No route exists

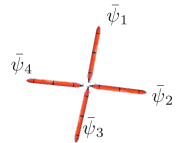
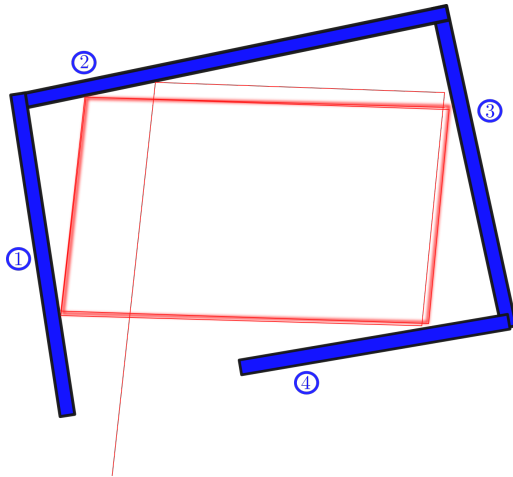


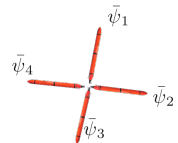
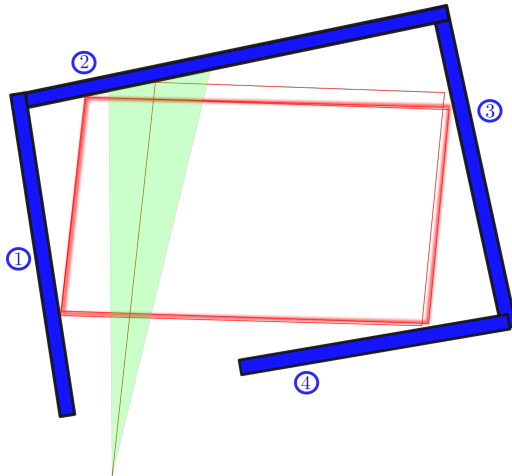


Contraction of the distance

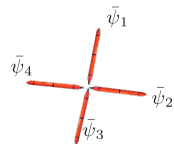
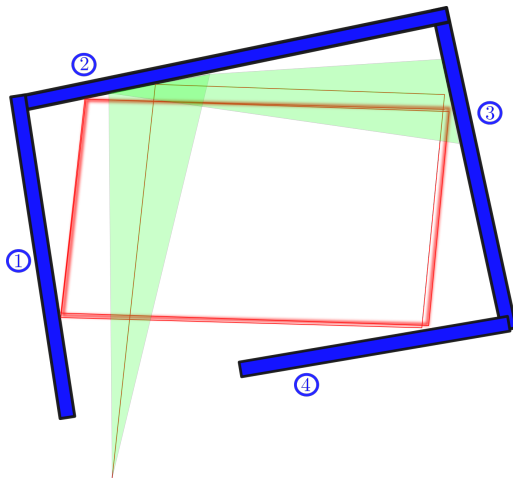


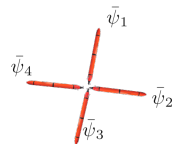
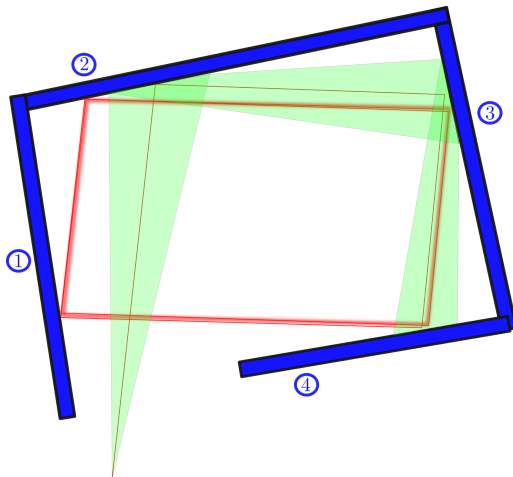
Contraction if  $\frac{\sin \beta}{\sin \alpha} < 1$

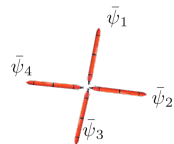
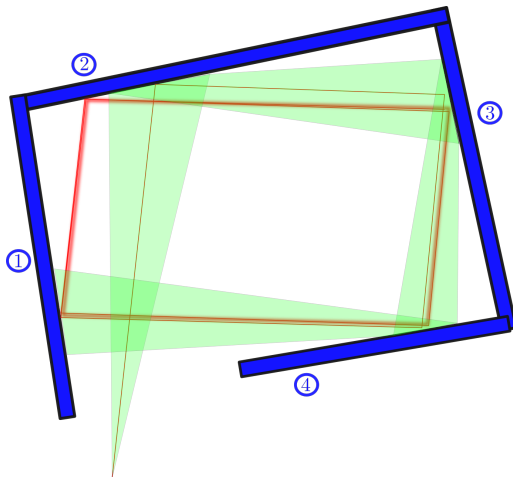


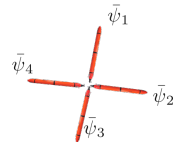
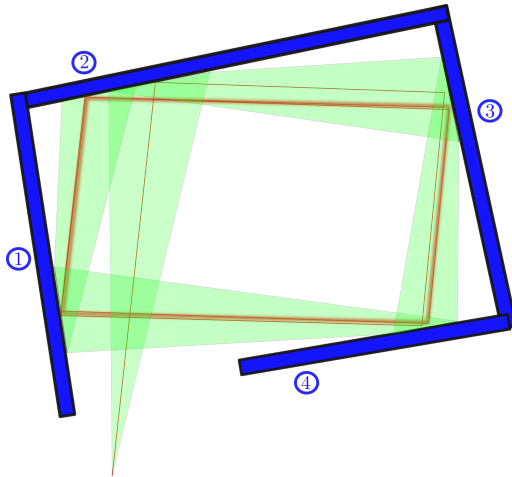


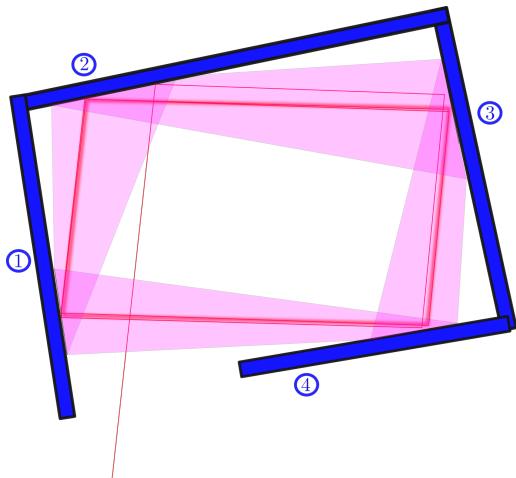


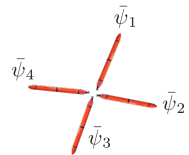
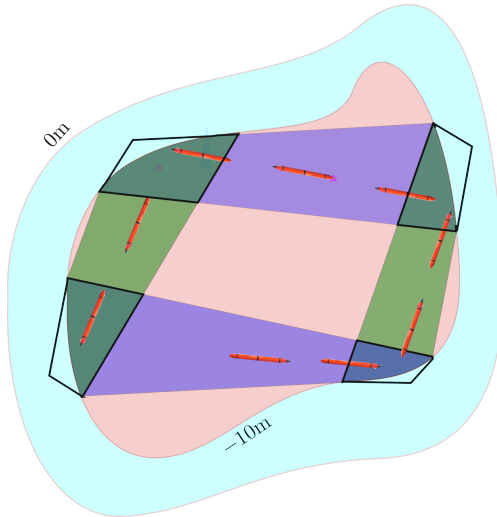


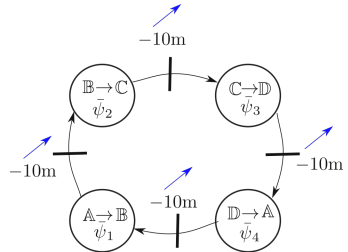
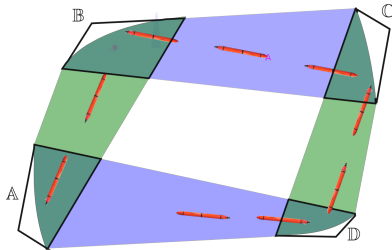


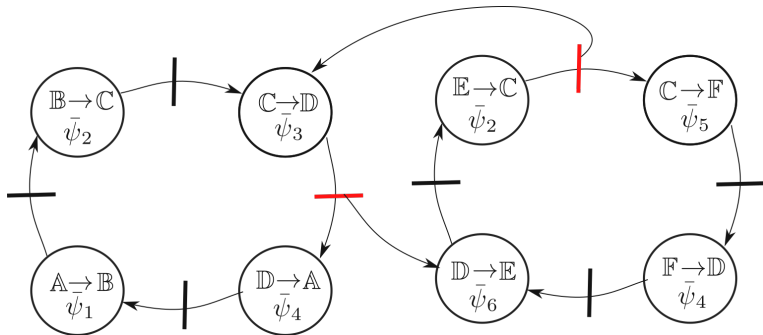






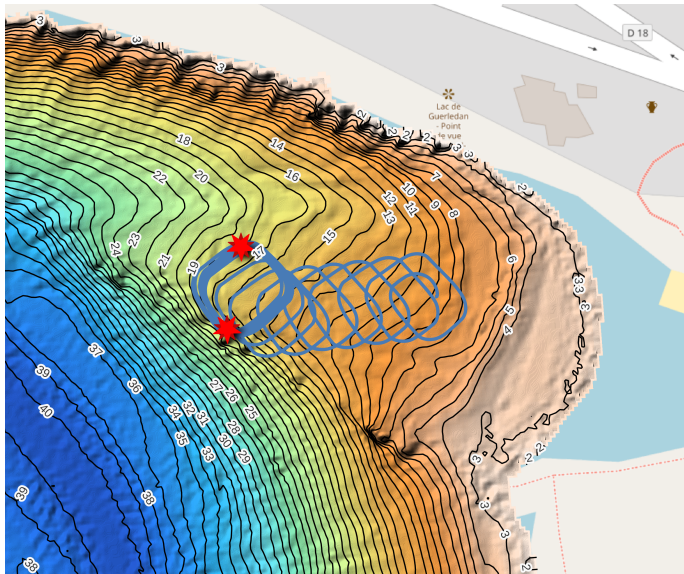








# Experiment (phd of Quentin Brateau)

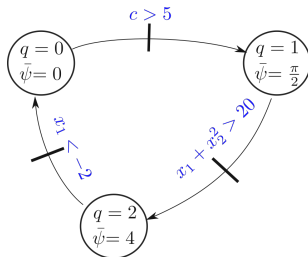


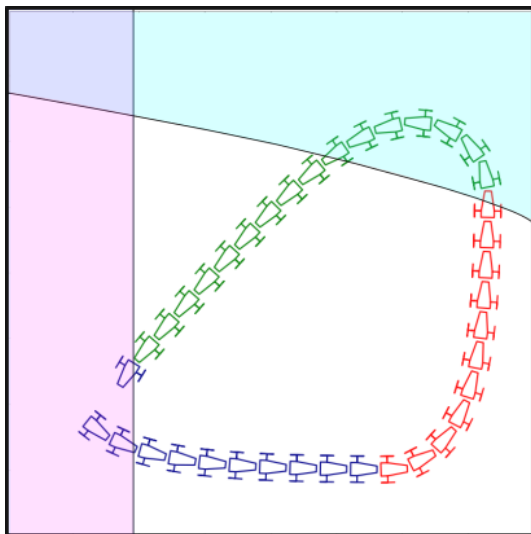
## 4. Proving the stability

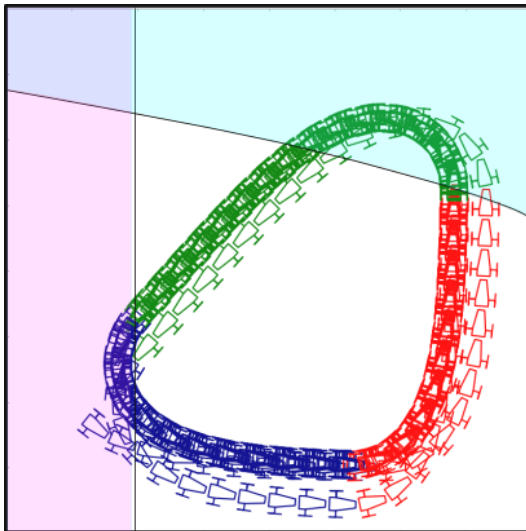
Consider the robot

$$\begin{cases} \dot{x}_1 &= \cos x_3 \\ \dot{x}_2 &= \sin x_3 \\ \dot{x}_3 &= u \end{cases}$$

with the heading control  $u = \sin(\bar{\psi} - x_3)$ .







# Interval arithmetic

$$[-1, 3] + [2, 5] = ?,$$

$$[-1, 3] \cdot [2, 5] = ?,$$

$$\text{abs}([-7, 1]) = ?$$



$$\begin{aligned}[-1, 3] + [2, 5] &= [1, 8], \\[-1, 3] \cdot [2, 5] &= [-5, 15], \\ \text{abs}([-7, 1]) &= [0, 7]\end{aligned}$$

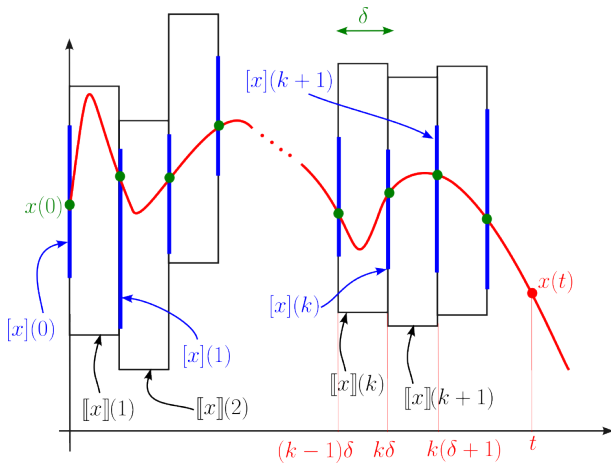
The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned} [f]([x_1], [x_2]) &= [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos[x_2] \\ &\quad + \sin[x_1] \cdot \sin[x_2] + 2. \end{aligned}$$

# Tubes



$$x(t) \in [x](t)$$

# Stability with Poincaré map

System:  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

How to prove that the system has a cycle ?

System:  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$

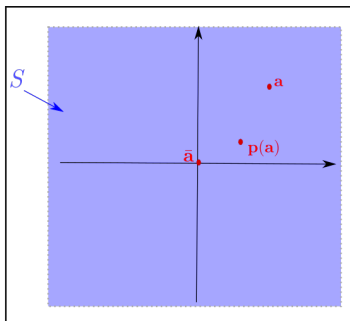
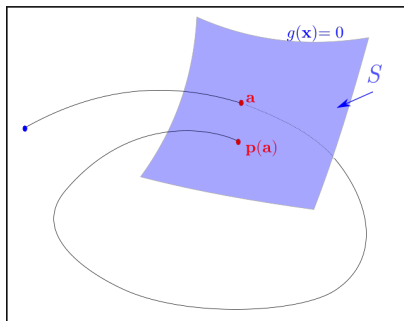
Poincaré section  $\mathcal{G}$ :  $g(\mathbf{x}) = 0$

We define

$$\mathbf{p}: \begin{array}{ccc} \mathcal{G} & \rightarrow & \mathcal{G} \\ \mathbf{a} & \mapsto & \mathbf{p}(\mathbf{a}) \end{array}$$

where  $\mathbf{p}(\mathbf{a})$  is the point of  $\mathcal{G}$  such that the trajectory initialized at  $\mathbf{a}$  intersects  $\mathcal{G}$  for the first time.





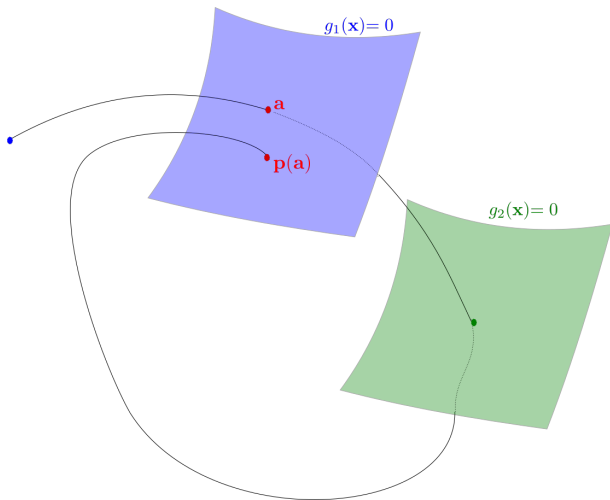
The Poincaré first recurrence map is defined by

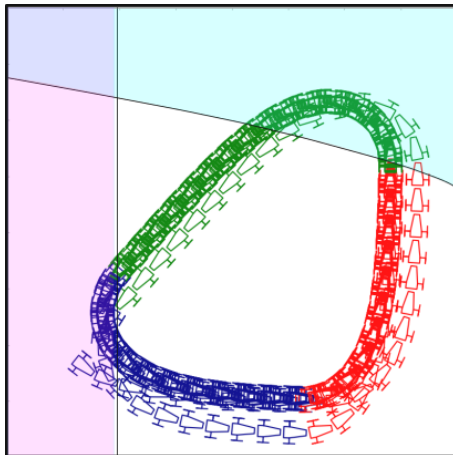
$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$

# With hybrid systems

Systems:  $\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}), i \in \{1, \dots, m\}$

Section  $i$ :  $g_i(\mathbf{x}) = 0$





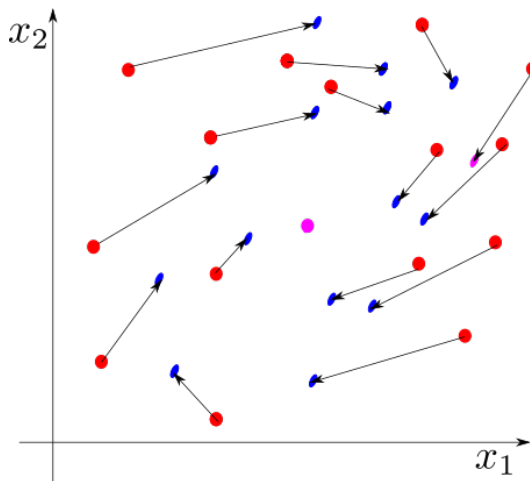
# Proving the stability

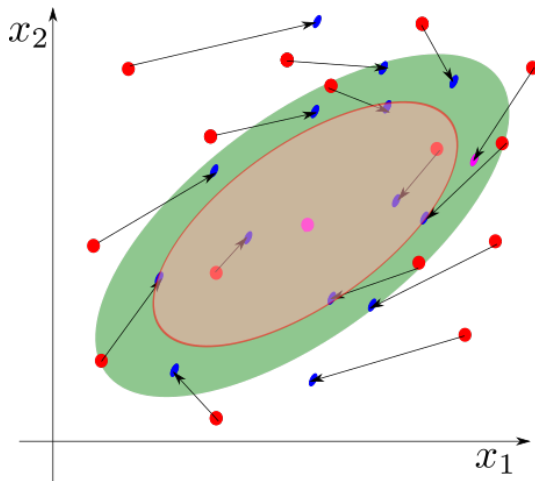
Consider the discrete time system

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$$

with  $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ .







We have to find

$$\mathcal{E}_{\mathbf{x}} : \mathbf{x}^T \cdot \mathbf{P} \cdot \mathbf{x} \leq \varepsilon$$

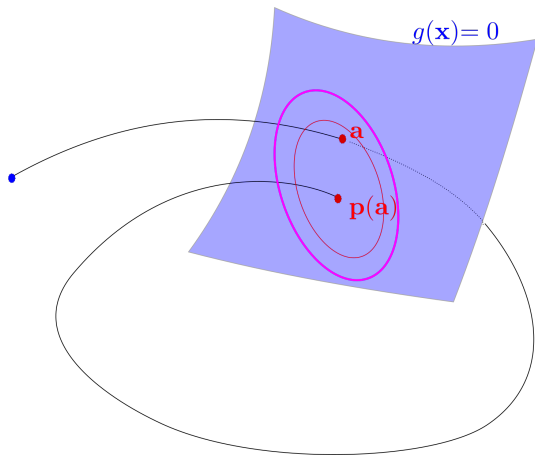
Such that

$$\mathbf{f}(\mathcal{E}_{\mathbf{x}}) \subset \mathcal{E}_{\mathbf{x}}$$

# Stability of cycles

The Poincaré first recurrence map is defined by

$$\mathbf{a}(k+1) = \mathbf{p}(\mathbf{a}(k))$$



# References

- 1 Interval and stability [3][13]
- 2 Route following [5][9]
- 3 Navigation with stable cycles [4]
- 4 Tubes [11][2]
- 5 Loops [12][1]
- 6 Integral formulation [6]
- 7 Ellipse and guaranteed integration [10]
- 8 Ellipses and guaranteed stability [8]
- 9 Axis-aligned Lyapunov equation [7]





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