Modeling and control of the rodwheel

Luc Jaulin

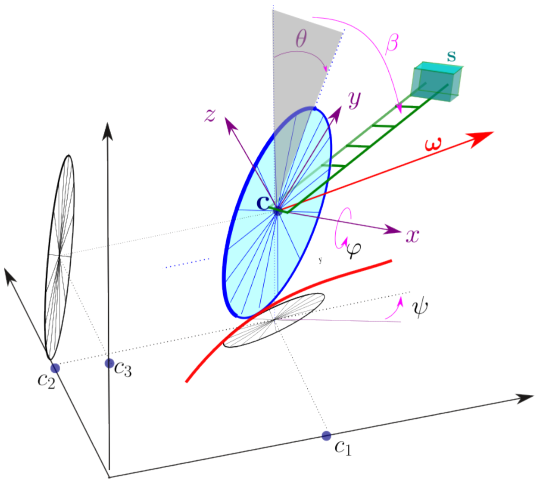
June 11, 2026, 4ème congrès annuel de la SAGIP, Bordeaux, France

1. Rodwheel

https://youtu.be/2KkKArsEH8Y

Autonomous wheel (Damien Esnault)

https://youtu.be/7VhmQ5LEGKs



The rodwheel

The state vector is $\mathbf{x} = (c_1, c_2, \varphi, \theta, \psi, \beta, \dot{\varphi}, \dot{\theta}, \dot{\psi}, \dot{\beta})$ where

- (c_1, c_2) is the vertical projection of center
- φ is the spin angle
- θ is the stand angle
- ψ is the heading,
- β is the rod angle

We want to find a state equations of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$$

2. Model

The Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = E_K - E_p$$

where E_K is the kinetic energy and E_p is the potential energy.

Denote by

$$\mathbf{q} = (c_1, c_2, \varphi, \theta, \psi, \beta)$$

the generalized coordinates of the system.

Using `sympy`, we get that the Lagrangian:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) =$$

$$\begin{gathered}
\tfrac{m}{8} r^2 (\dot\theta \sin\varphi - \dot\psi \cos\theta \cos\varphi)^2 + \tfrac{m}{8} r^2 (\dot\theta \cos\varphi + \dot\psi \sin\varphi \cos\theta)^2 \\
+ \tfrac{m}{4} r^2 (\dot\varphi - \dot\psi \sin\theta)^2 + \tfrac{m}{2} (r^2 \dot\theta^2 \sin^2\theta + \dot c_1^2 + \dot c_2^2) - \mu(\ell \cos\beta + r) \cos\theta \\
+ \frac{\mu \left(\ell \sin\psi \cos\beta \cos\theta \dot\theta + \ell (\sin\beta \sin\psi + \sin\theta \cos\beta \cos\psi) \dot\psi - \ell (\sin\beta \sin\theta \sin\psi + \cos\beta \cos\psi) \dot\beta + \dot c_2 \right)^2}{2} \\
+ \frac{\mu \left(\ell \cos\beta \cos\theta \cos\psi \dot\theta + \ell (\sin\beta \cos\psi - \sin\theta \sin\psi \cos\beta) \dot\psi + \ell (\sin\psi \cos\beta - \sin\beta \sin\theta \cos\psi) \dot\beta + \dot c_1 \right)^2}{2} \\
+ \tfrac{\mu}{2} \left(\ell \sin\beta \cos\theta \dot\beta + (\ell \sin\theta \cos\beta + r \sin\theta) \dot\theta \right)^2 \\
- gmr \cos\theta
\end{gathered}$$

The evolution of $\mathbf{q}$ obeys to the Euler-Lagrange equations:

$$\underbrace{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}}_{\mathcal{Q}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})} = \underbrace{\begin{pmatrix} \tau_{\mathbf{c}_1} \\ \tau_{\mathbf{c}_2} \\ \tau_{\varphi} \\ \tau_{\theta} \\ \tau_{\psi} \\ \tau_{\beta} \end{pmatrix}}_{\tau} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \cdot u$$

The variables $(\mathbf{q},\dot{\mathbf{q}})$ are linked by non holonomic constraints

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = r\begin{pmatrix} \sin\psi & \cos\psi\cos\theta & -\sin\psi\sin\theta \\ -\cos\psi & \sin\psi\cos\theta & \cos\psi\sin\theta \end{pmatrix}\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

D'Alembert's principle: *for arbitrary virtual displacements, the constraint forces don't do any work.*

Therefore:

$$\begin{pmatrix} \tau_{c_1} \\ \tau_{c_2} \\ \tau_{\varphi} \\ \tau_{\theta} \\ \tau_{\psi} \\ \tau_{\beta} \end{pmatrix} = \mathbf{A}^{\mathsf{T}}(\mathbf{q}) \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

where

$$\mathbf{A}(\mathbf{q}) = \begin{pmatrix} 1 & 0 & -r\sin\psi & -r\cos\psi\cos\theta & r\sin\psi\sin\theta & 0 \\ 0 & 1 & r\cos\psi & -r\sin\psi\cos\theta & -r\cos\psi\sin\theta & 0 \end{pmatrix}.$$

The λ_i's are the *Lagrange parameters*.

The state equations of the rodwheel are

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = r \begin{pmatrix} \sin\psi & \cos\psi\cos\theta & -\sin\psi\sin\theta \\ -\cos\psi & \sin\psi\cos\theta & \cos\psi\sin\theta \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\beta} \end{pmatrix}$$

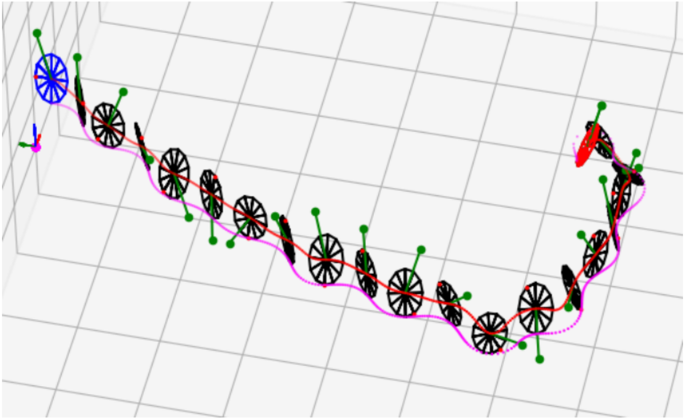
$$\begin{pmatrix} \ddot{\varphi} \\ \ddot{\theta} \\ \ddot{\psi} \\ \ddot{\beta} \end{pmatrix} = (\mathbf{0}_4\ \mathbf{I}_4) \cdot \left(\begin{pmatrix} \frac{\partial \mathcal{S}}{\partial \lambda} & \frac{\partial \mathcal{S}}{\partial \ddot{\mathbf{q}}} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right)^{-1} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \cdot u - \mathcal{S} \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right)$$

where

$$\mathcal{S}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\lambda) = \left(\begin{array}{c} \frac{\partial \mathbf{a}(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial \mathbf{a}(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} \cdot \ddot{\mathbf{q}} \\ \mathcal{Q}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) - \mathbf{A}^{\mathsf{T}}(\mathbf{q}) \left(\begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right) \end{array} \right)$$

$$\mathbf{a}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{A}(\mathbf{q}) \cdot \dot{\mathbf{q}}$$

$$\mathcal{Q}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$



Evolution of rodwheel with $u = 0$

3. Control

We want the rodwheel

- goes straight,
- the rod upward,
- at a given speed $\bar{v}$,
- at a given heading.

Several difficulties have to be raised:

- We have a single input u for several state variables to control,
- The system is not controllable once the rodwheel goes straight,
- The system is unstable with respect to the rod angle β,
- The system is unstable with respect to the stand angle θ.

Consider the situation where the stand angle $\theta(t) = 0$.
Take the controller:

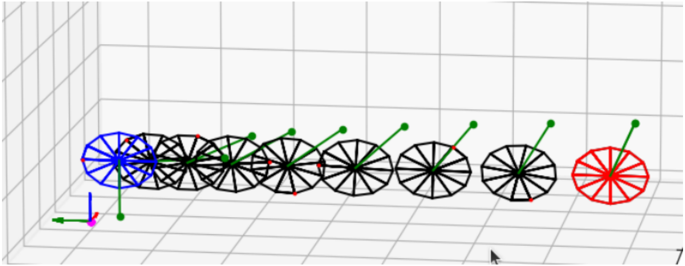
$$u = k_1(\beta - \beta_0) + k_2\dot{\beta}$$

where $\beta - \beta_0$ is the error.
We can control rod angle.

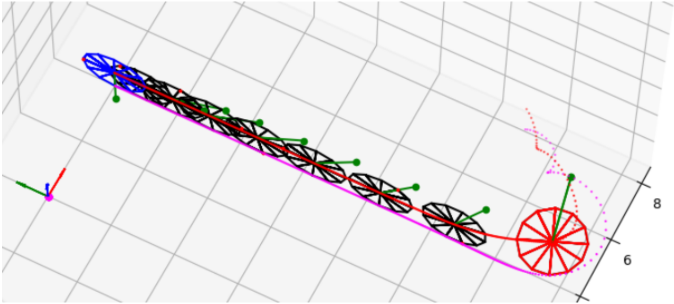
Consider again the situation $\theta(t) = 0$.
Take the controller:

$$\begin{aligned} u &= k_1(\beta - \beta_0) + k_2\dot{\beta} \\ \beta_0 &= k_3 \tanh(\bar{v} - \dot{\varphi}) \end{aligned}$$

We can control the speed of the rodwheel.



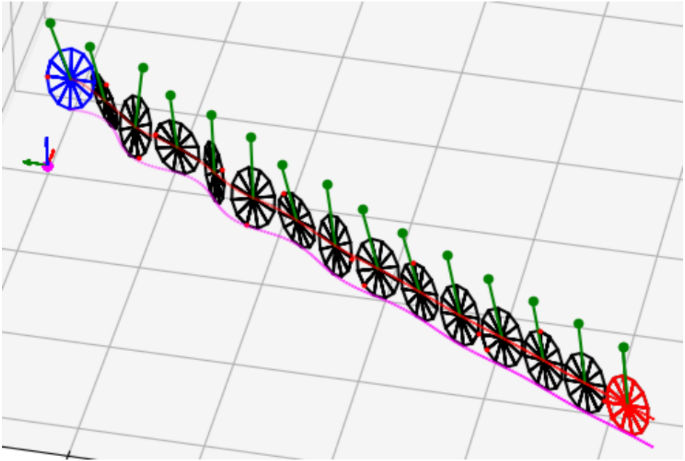
Rodwheel with $\theta(t) = 0$ which accelerates to reach the speed $\bar{v}$



A tiny perturbation with respect to θ generates a fall

To stabilize θ to 0, we add a term which accelerates when θ deviates from zero. We get the following controller:

$$\begin{aligned} u &= k_1(\beta - \beta_0) + k_2\dot{\beta} + k_4|\theta| \\ \beta_0 &= k_3 \tanh(\bar{v} - \dot{\varphi}) \end{aligned}$$



The controller stabilizes the precession of the rodwheel

References

1. Realization of the rodwheel [1]
2. Model and control of the rodwheel [2][3]
3. Source code https://webperso.ensta.fr/jaulin/rodwheel.html

D. Esnault.
Robot d'engagement gyrostabilisé de reconnaissance.
Rapport de PFE, ENSTA-Bretagne, Brest, France, 2023.

L. Jaulin.
Modeling and control of the rodwheel.
arXiv:2401.08666, math.NA, 2024.

L. Jaulin.
Modeling and control of an hoverboard.
hal-04953210, 2025.