Swim with brackets

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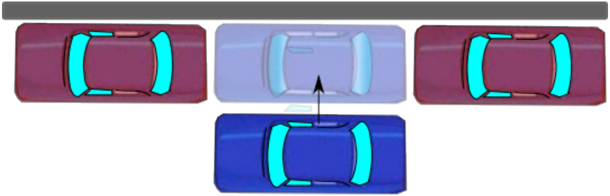
1. Control with Lie brackets



To park, the blue car needs to move sideway

$$\begin{cases} \dot{x}_1 = u_1 \cos x_3 \\ \dot{x}_2 = u_1 \sin x_3 \\ \dot{x}_3 = u_2 \end{cases}$$

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u_2$$

The Lie bracket between the two vector fields $\mathbf{f}$ and $\mathbf{g}$ is

$$[\mathbf{f},\mathbf{g}] = \frac{d\mathbf{g}}{d\mathbf{x}} \cdot \mathbf{f} - \frac{d\mathbf{f}}{d\mathbf{x}} \cdot \mathbf{g}.$$

We get

$$\left[\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

Consider the driftless system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \cdot u_1 + \mathbf{g}(\mathbf{x}) \cdot u_2.$$

Take a periodic input and zero mean input $\mathbf{u} = (u_1, u_2)$. Define

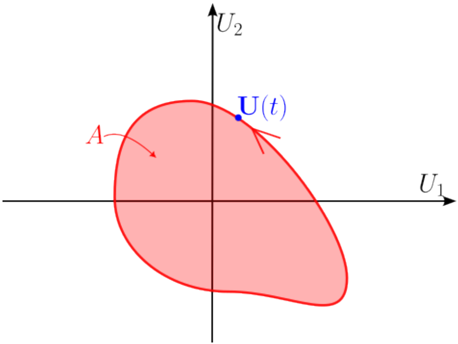
$$\mathbf{U}(t) = \int_0^t \mathbf{u}(\tau) d\tau$$

If the period δ is small, the displacement after one period is given by:

$$[\mathbf{f}, \mathbf{g}] \cdot A$$

where

$$A = \text{Area}(\mathbf{U})$$


U_2
$\mathbf{U}(t)$
A
U_1

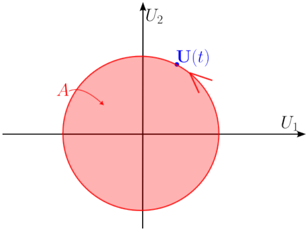
This means that

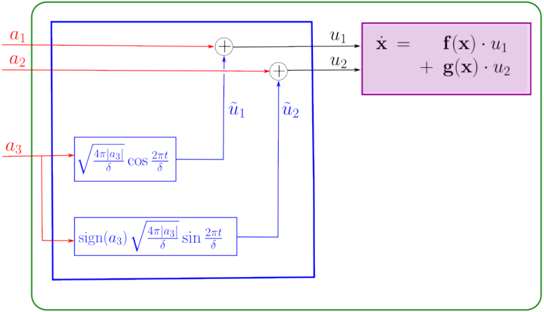
$$\mathbf{x}(t+\delta)-\mathbf{x}(t)=[\mathbf{f},\mathbf{g}](\mathbf{x})\cdot A+o(\delta^2).$$

Example.

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \sqrt{\frac{4\pi a_3}{\delta}} \begin{pmatrix} \cos(2\pi\frac{t}{\delta}) \\ \sin(2\pi\frac{t}{\delta}) \end{pmatrix}$$

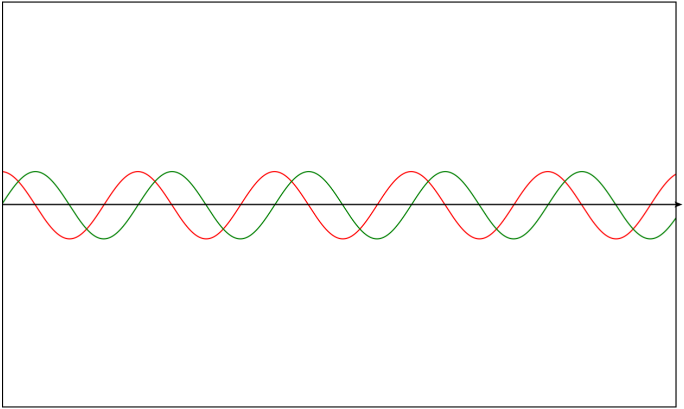
where a_3 is some parameter.



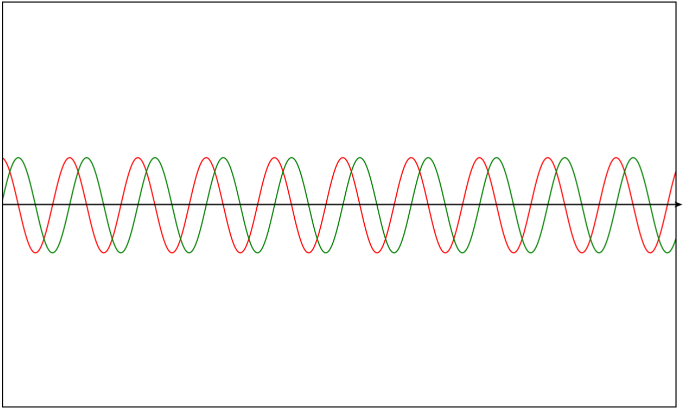


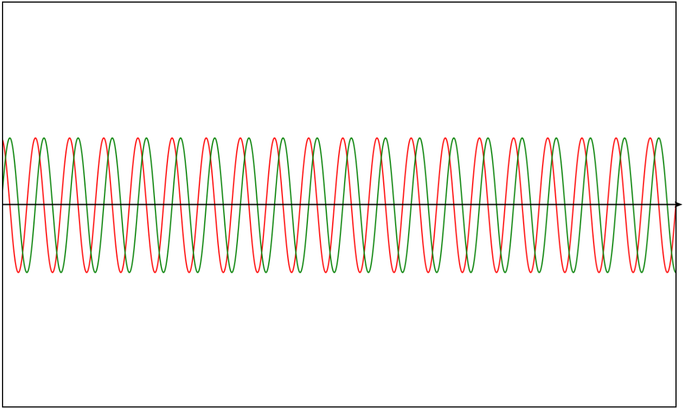
$\Leftrightarrow$

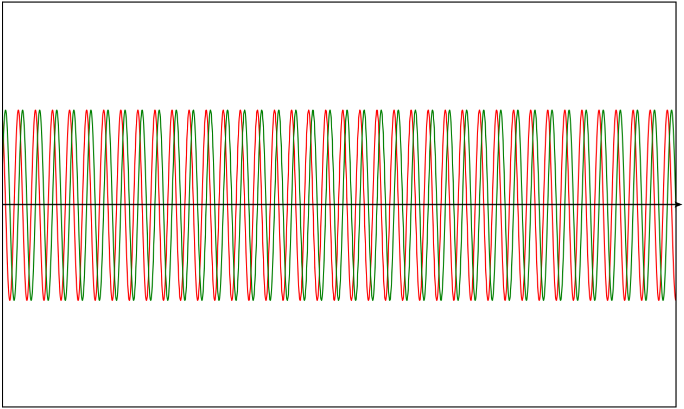
$$\begin{aligned}\dot{\mathbf{x}} = \; & \mathbf{f}(\mathbf{x})\cdot a_1 \\ + \; & \mathbf{g}(\mathbf{x})\cdot a_2 \\ + \; & [\mathbf{f},\mathbf{g}](\mathbf{x})\cdot a_3\end{aligned}$$

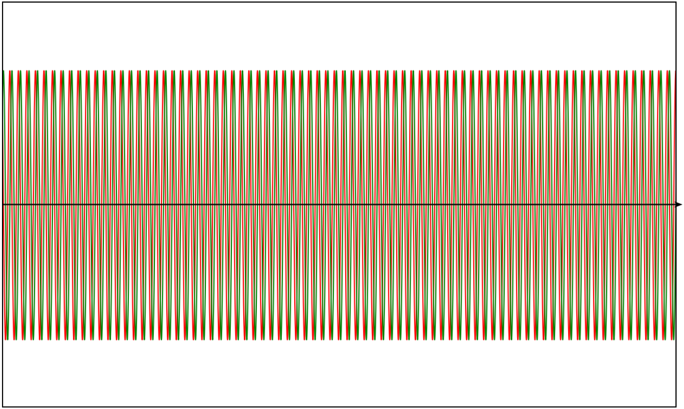


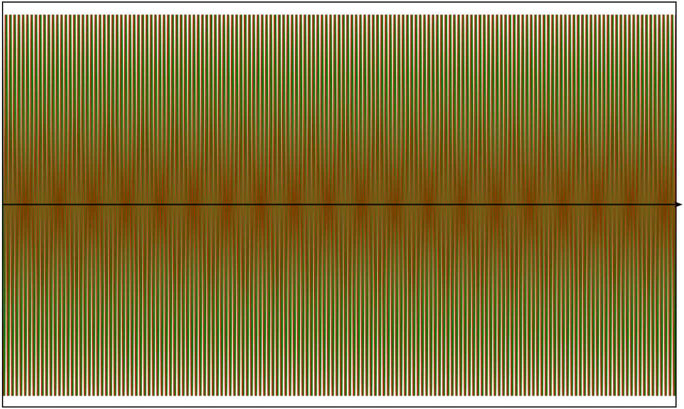
$u_1(t)$ and $u_2(t)$











We converge to a *vortex* "distribution" $\mathbb{R} \to \mathbb{R}^2$

Remark. Since $\delta = o(1)$, we cannot go fast along $[\mathbf{f}, \mathbf{g}]$.

For our Dubins car:

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot u_2$$

We have

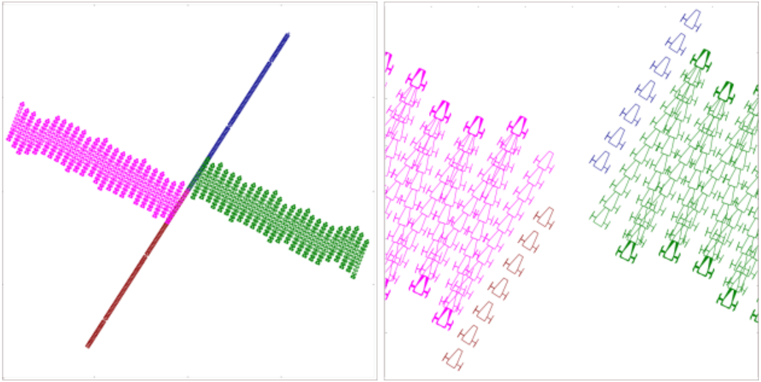
$$[\mathbf{f},\mathbf{g}](\mathbf{x}) = \underbrace{\frac{d\mathbf{g}}{d\mathbf{x}}(\mathbf{x})}_{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underbrace{\mathbf{f}(\mathbf{x})}_{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}} - \underbrace{\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x})}_{\begin{pmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{pmatrix}} \cdot \underbrace{\mathbf{g}(\mathbf{x})}_{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$= \begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix}$$

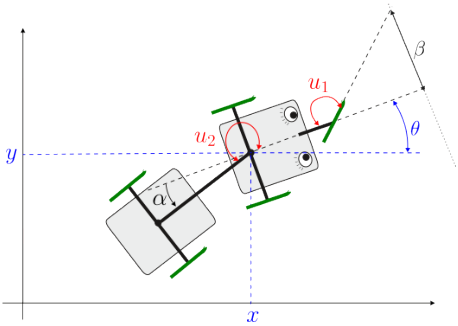
We can now move the car laterally.

If we apply the cyclic sequence, we get

$$\dot{\mathbf{x}} = \underbrace{\begin{pmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} \cdot a_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}(\mathbf{x})} \cdot a_2 + \underbrace{\begin{pmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{pmatrix}}_{[\mathbf{f},\mathbf{g}](\mathbf{x})} \cdot a_3$$



2. Skate car


β
u_1
u_2
θ
y
α
x

Model for the skate car

$$\left\{\begin{array}{lcl} \dot{x} & = & v\cos\theta \\ \dot{y} & = & v\sin\theta \\ \dot{\theta} & = & v\beta \\ \dot{v} & = & -(\beta+\sin\alpha)\,u_2 \\ \dot{\alpha} & = & -v\,(\beta+\sin\alpha) \\ \dot{\beta} & = & u_1 \end{array}\right.$$

Velocity model

$$\begin{cases} \dot{\theta} & = & v\beta \\ \dot{v} & = & -(\beta + \sin\alpha)\, u_2 \\ \dot{\alpha} & = & -v\,(\beta + \sin\alpha) \\ \dot{\beta} & = & u_1 \end{cases}$$

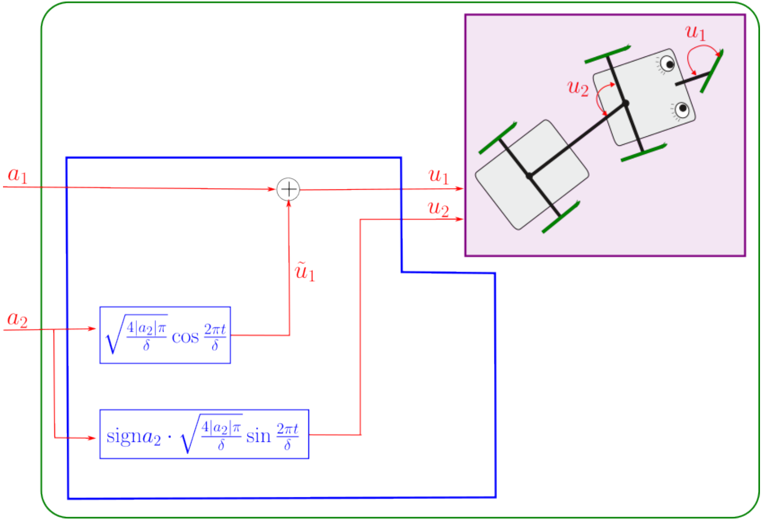
$$\begin{pmatrix} \dot{\theta} \\ \dot{v} \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} v\beta \\ 0 \\ -v(\beta + \sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot u_1 + \underbrace{\begin{pmatrix} 0 \\ -\beta - \sin\alpha \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{g}_2} \cdot u_2$$

We have

$$
\begin{aligned}
[\mathbf{g}_1, \mathbf{g}_2] &= \frac{d\mathbf{g}_2}{d\mathbf{x}} \cdot \mathbf{g}_1 - \underbrace{\frac{d\mathbf{g}_1}{d\mathbf{x}}}_{=\mathbf{0}} \cdot \mathbf{g}_2 \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos\alpha & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

We take the controller

$$\mathbf{u}(t) = \begin{pmatrix} \sqrt{\frac{4|a_2|\pi}{\delta}} \cos \frac{2\pi t}{\delta} + a_1 \\ \mathsf{sign} a_2 \cdot \sqrt{\frac{4|a_2|\pi}{\delta}} \sin \frac{2\pi t}{\delta} \end{pmatrix}$$


a_1
a_2
u_1
u_2
$\tilde{u}_1$
$\sqrt{\frac{4|a_2|\pi}{\delta}} \cos \frac{2\pi t}{\delta}$
$\text{sign} a_2 \cdot \sqrt{\frac{4|a_2|\pi}{\delta}} \sin \frac{2\pi t}{\delta}$
u_2
u_1

We now have the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\delta} \\ \dot{\beta} \end{pmatrix} = \underbrace{\begin{pmatrix} v\cos\theta \\ v\sin\theta \\ v\beta \\ 0 \\ -v(\beta+\sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot a_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{[\mathbf{g}_1,\mathbf{g}_2]} \cdot a_2$$

Assume that we want to control v and β. i.e.

$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{a}$$

We want

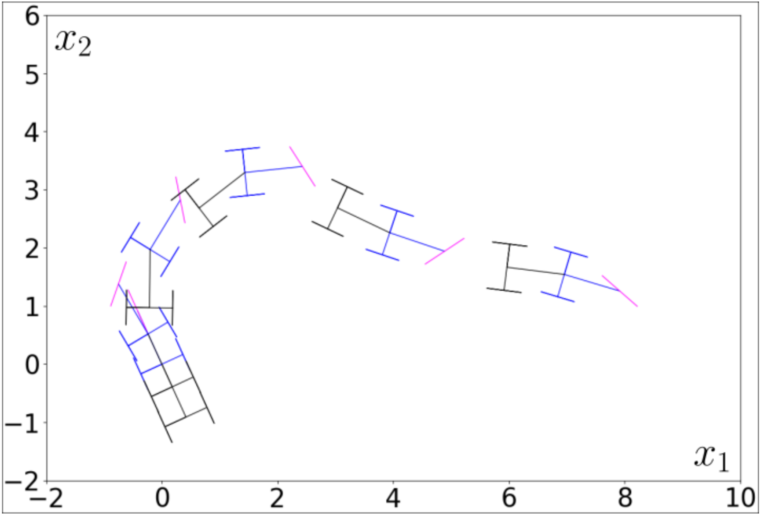
$$\begin{pmatrix} \dot{v} \\ \dot{\beta} \end{pmatrix} = K \begin{pmatrix} \bar{v} - v \\ \bar{\beta} - \beta \end{pmatrix}$$

Thus we take

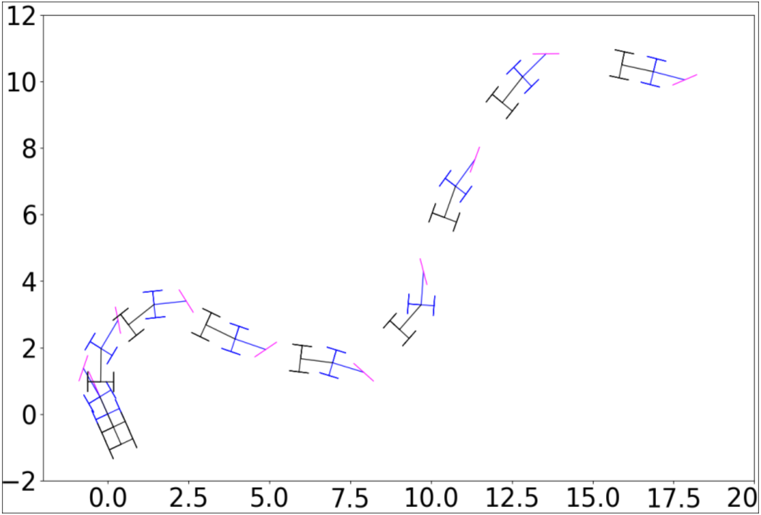
$$\mathbf{a} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \left(K \begin{pmatrix} \bar{v} - v \\ \bar{\beta} - \beta \end{pmatrix} \right)$$

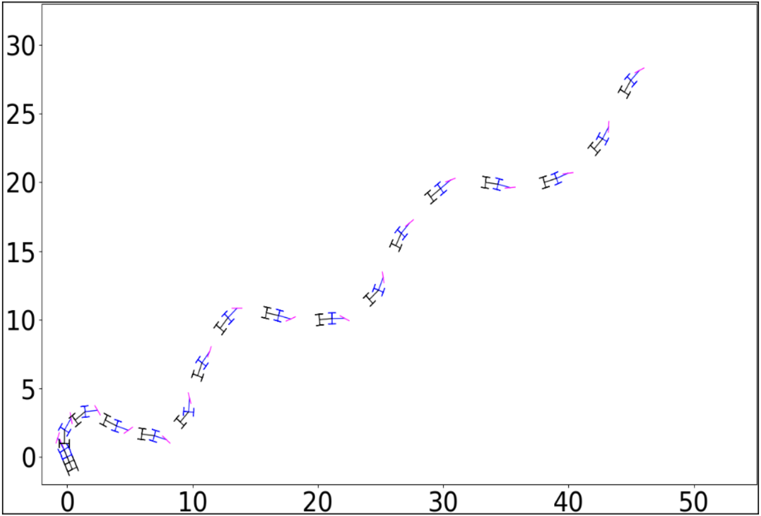
To have a heading control, we take

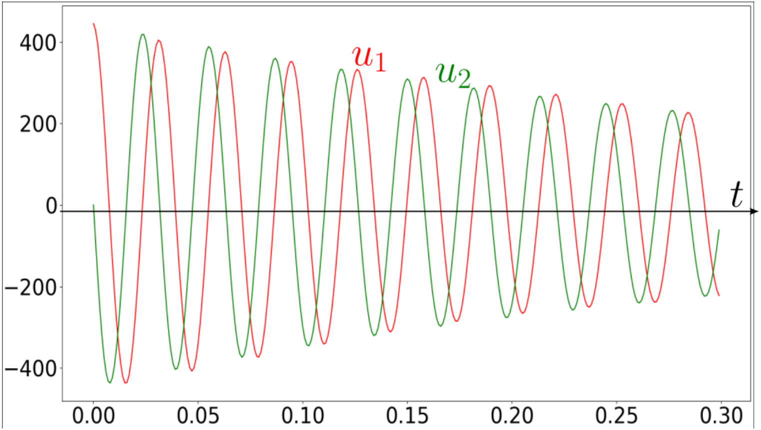
$$\dot{\bar{\beta}} = -0.01\bar{\beta} + 0.1(\bar{\theta} - \theta).$$

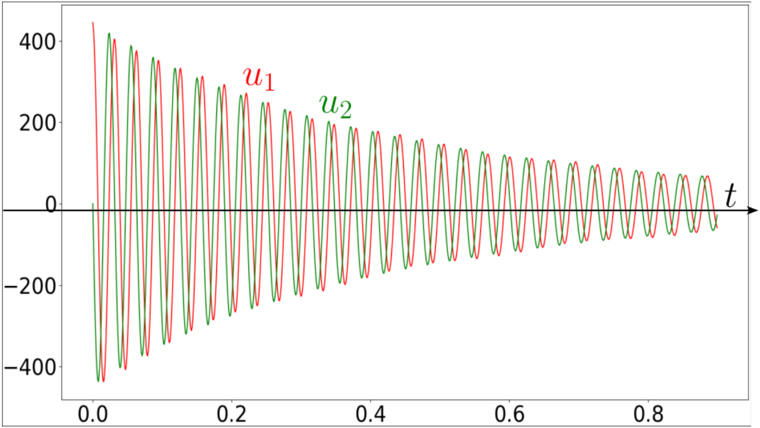


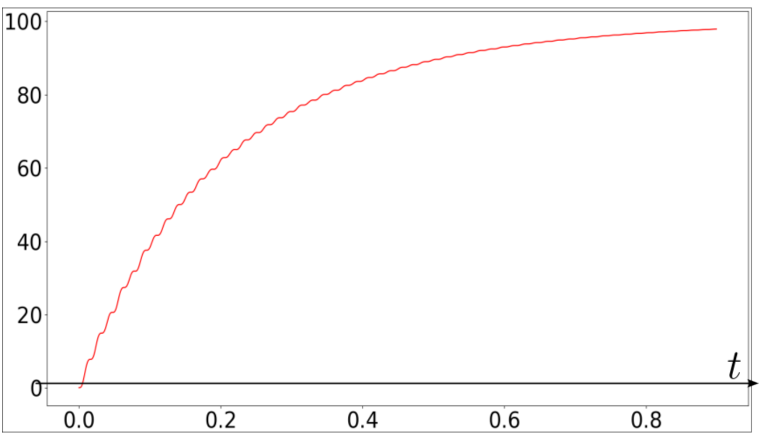
Desired heading $\bar{\theta} = \frac{\pi}{6}$ and a desired speed $\bar{v} = 100$


12
10
8
6
4
2
0
−2
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20


30
25
20
15
10
5
0
0
10
20
30
40
50


400
200
0
−200
−400
0.00
0.05
0.10
0.15
0.20
0.25
0.30
u_1
u_2
t


400
200
0
−200
−400
0.0
0.2
0.4
0.6
0.8
u_1
u_2
t


100
80
60
40
20
0
t
0.0
0.2
0.4
0.6
0.8

3. Abstract swimmer

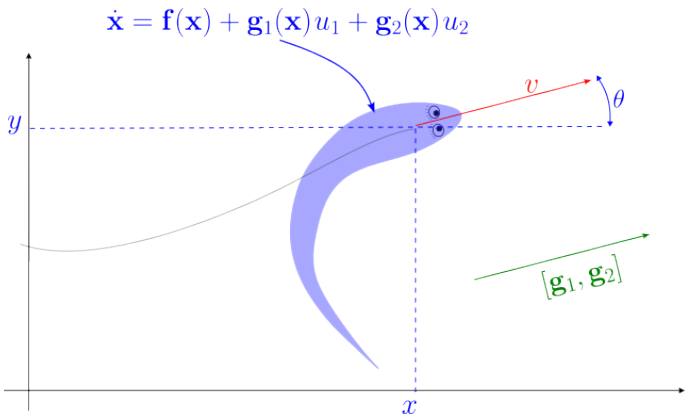
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x}) \cdot u_1 + \mathbf{g}_2(\mathbf{x}) \cdot u_2$$

with

$$\mathbf{x} = (v, \cdots)$$

v almost driftless, *i.e.*,$f_1(\mathbf{x}) \simeq 0$

$$[\mathbf{g}_1, \mathbf{g}_2] \parallel (1, 0, 0, \dots)$$


$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2$
v
θ
y
x
$[\mathbf{g}_1, \mathbf{g}_2]$

The skate car:

$$\begin{pmatrix} \dot v \\ \dot x \\ \dot y \\ \dot\theta \\ \dot\alpha \\ \dot\beta \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ v\cos\theta \\ v\sin\theta \\ v\beta \\ -v\,(\beta+\sin\alpha) \\ 0 \end{pmatrix}}_{\mathbf{f}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{g}_1} \cdot u_1 + \underbrace{\begin{pmatrix} -(\beta+\sin\alpha) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{g}_2} \cdot u_2$$

with

$$[\mathbf{g}_1, \mathbf{g}_2] = (-1, 0, 0, \dots)$$

References

1. Lie bracket control [4] [3] [5]
2. Skate car [2]
3. Swimming robots [7][6][1]
4. Source code: https://webperso.ensta.fr/jaulin/snake.html

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